DISCRETE MATHEMATICS AND APPLICATIONS

Logic 2

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Chapter Description

- Chapter Outline

  3.4 Predicates and Quantifiers

  3.5 Rules of Inference

- Aims
  - Find the truth values of predicates and quantifiers
  - Apply modus ponens and modus tollens in identifying valid arguments

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References

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Predicate

Predicate is essentially a sentence that gives information about the subject.

subject predicate

Grammar

- Yusuf is a computer science student at UMP.

variable predicate

Logic

- x is a computer science student at UMP.

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The Truth Set of $P(x)$

If $P(x)$ is a predicate and x has domain D, the truth set of $P(x)$ is the set of all elements of D that make $P(x)$ true when they are substituted for x.

The truth set of $P(x)$ is denoted by:

$$\{x \in D \mid P(x)\}$$

The set of all x such that

We read as "the set of all x in D such that $P(x)$" .

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The Truth Set of $P(x)$: Example

Let $Q(n)$ be the predicate "n is factor of 6". Find the truth set of $Q(n)$ if:

(a) the domain of n is the set Z^+ of all positive integers

(b) the domain of n is the set Z of all integers.

(a) The truth set is {1, 2, 3, 6} because these are exactly the positive integers that divides 6 evenly.

(b) The truth set is {1, 2, 3, 6, -1, -2, -3, -6} because the negative integers -1, -2, -3, and -6 also divide into 6 without leaving a remainder.

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Quantifier

Quantifiers

Universal

A predicate is TRUE for every element.

Existential

There is at least one element under consideration for which the predicate is TRUE.

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Universal Quantifier

- Symbol of universal quantifier, $\forall$.
- Read as "for all".
- Can also be read as "for any"," for every" or "for each".

- The universal quantification of $Q(x)$ is the statement

  "$Q(x)$ for all values of x in the domain (D)"

$$\forall x \in D, Q(x) \quad \text{or} \quad \forall x Q(x)$$

- True if and only if every x in D is true.
- False if and only if $Q(x)$ is false for at least one x in D.

- Counterexample : A value of x for which $Q(x)$ is false.

Universal Quantifier: Example 1

Let D = {1, 2, 3, 4, 5}.

Show that this statement is true.

Solution:

Check that "$x^2 \geq x$" is true for each individual x in D.

$$1^2 \geq 1, \qquad 2^2 \geq 2, \qquad 3^2 \geq 3, \qquad 4^2 \geq 4, \qquad 5^2 \geq 5$$

Hence "$\forall x \in D, x^2 \geq x$" is true.

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Universal Quantifier: Example 2

Consider the statement

$$\forall x \in \mathbb{R}, x^2 \geq x$$

Determine the counterexample to show that the statement is false.

Solution:

Counterexample: Take $x = \frac{1}{2}$. Then x is in $\mathbb{R}$ (since $\frac{1}{2}$ is a real number) and $\left(\frac{1}{2}\right)^2 = \frac{1}{4} \not\geq \frac{1}{2}$. Hence "$\forall x \in \mathbb{R}, x^2 \geq x$" is false.

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Existential Quantifier

- Symbol of existential quantifier, $\exists$.
- Read as "there exist".
- Can also be read as "there is at least one" or "for some".

- The existential quantification of $Q(x)$ is the statement

  " There exists an element x in the domain (D) such that $Q(x)$ "

  $\exists x \in D, Q(x)$ or $\exists x Q(x)$

- True if and only if at least one x in D is true.
- False if and only if $Q(x)$ is false for all x in D.

- Counterexample : A value of x for which $Q(x)$ is false.

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Existential Quantifier: Example 1

Consider the statement

$$\exists m \in \mathbb{Z}, m^2 = m$$

Show that this statement is true.

Solution:

Observe that $1^2 = 1$. Thus "$m^2 = m$" is true for at least one integer m. Hence "$\exists m \in \mathbb{Z}$ such that $m^2 = m$" is true.

Existential Quantifier: Example 2

Let $E = \{5, 6, 7, 8, 9, 10\}$ and consider the statement

$$\exists m \in E, m^2 = m$$

Show that this statement is false.

Solution:

Note that $m^2 = m$ is not true for any integers m from 5 to 10.

$$5^2 = 25 \neq 5,\ 6^2 = 36 \neq 6,\ 7^2 = 49 \neq 7,\ 8^2 = 64 \neq 8,\ 9^2 = 81 \neq 9,\ 10^2 = 100 \neq 10.$$

Thus, "$\exists m \in E$ such that $m^2 = m$" is false.

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Combining quantifiers

For every x in $\mathbb{Z}$, there is y in $\mathbb{Z}$, so that $x + y = 0$

In symbols

$$\forall x \in \mathbb{Z}, \exists y \in \mathbb{Z}, x + y = 0$$

This is a true statement. Why?

This sentence makes a claim about an arbitrary integer x. It says that no matter what x is, something is true, we can find an integer y so that $x + y = 0$.

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Combining quantifiers: Example

Determine whether the following sentences in integers either true or false.

$\forall x, \forall y, x+y=0$ **False**

$\exists x, \forall y, x+y=0$ **False**

$\exists x, \exists y, x+y=0$ **True**

$\forall x, \forall y, xy=0$ **False**

$\forall x, \exists y, xy=0$ **True**

$\exists x, \forall y, xy=0$ **True**

$\exists x, \exists y, xy=0$ **True**

Argument (i)

- Argument – sequence of propositions that end with a conclusion.

$$\begin{array}{rl} & \left.\begin{array}{l} p_1 \\ p_2 \\ p_3 \\ \cdot \\ \cdot \\ \cdot \\ p_n \end{array}\right\} \text{premises} \\ \hline \therefore & \left. q \right\} \text{conclusion} \end{array}$$

- The argument is valid if and only if all the premises are true and conclusion also must be true.

Argument (ii)

Consider the argument:

If Yusuf is a handsome boy, then he is an engineer.

Yusuf is a handsome boy.

$\therefore$ Yusuf is an engineer.

Then, the argument form:

$$\begin{array}{ll} & p \to q \\ & p \\ \hline \therefore & q \end{array}$$

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Valid and Invalid Argument: Example 1 (i)

- Determine whether the following argument is valid or invalid.

$$
\begin{array}{ll}
 & p \rightarrow r \\
 & \sim q \vee \sim r \\
 & p \rightarrow q \\
\hline
\therefore & r
\end{array}
$$

- How to determine whether the argument is valid or invalid?

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Valid and Invalid Argument: Example 1 (ii)

p	q	r	$p \to r$	$\sim q$	$\sim r$	$\sim q \vee \sim r$	$p \to q$	r
T	T	T	T	F	F	F	T	T
T	T	F	F	F	T	T	T	F
T	F	T	T	T	F	T	F	T
T	F	F	F	T	T	T	F	F
F	T	T	T	F	F	F	T	T
F	T	F	T	F	T	T	T	F
F	F	T	T	T	F	T	T	T
F	F	F	T	T	T	T	T	F

Critical row: to check the validity of an argument

Critical rows

$\therefore$ The argument is invalid.

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Valid and Invalid Argument: Example 2 (i)

- Show that the following argument is invalid.

$$\begin{array}{l} \quad q \to r \\ \quad p \to q \\ \quad \sim q \vee r \\ \hline \therefore \; q \end{array}$$

- How to show the argument is invalid?

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Valid and Invalid Argument: Example 2 (ii)

p	q	r	$q \to r$	$p \to q$	$\sim q$	$\sim q \vee r$	q
T	T	T	T	T	F	T	T
T	T	F	F	T	F	F	T
T	F	T	T	F	T	T	F
T	F	F	T	F	T	T	F
F	T	T	T	T	F	T	T
F	T	F	F	T	F	F	T
F	F	T	T	T	T	T	F
F	F	F	T	T	T	T	F

Critical row: to check the validity of an argument

Critical rows

∴ It is shown that the argument is invalid.

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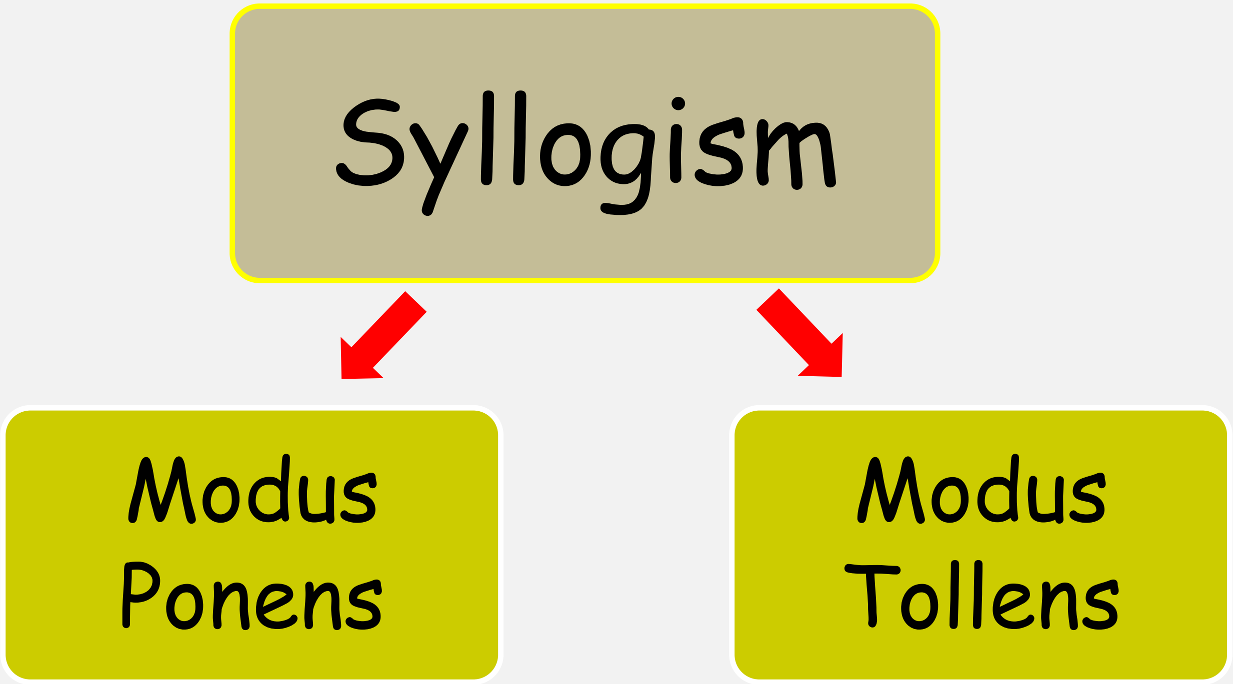
Syllogism

Syllogism - an argument consisting of two premises and a conclusion.



Modus ponens

- Modus Ponens: method of affirming.

The argument form:

$$\begin{array}{ll} & p \to q \\ & p \\ \hline \therefore & q \end{array}$$

p	q	$p \to q$	p	q
T	T	T	T	T
T	F	F	T	F
F	T	T	F	T
F	F	T	F	F

$\therefore$ The argument is valid.

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Modus tollens

- Modus Tollens: method of denying.

The argument form:

$$\begin{array}{ll} & p \to q \\ & \sim q \\ \hline \therefore & \sim p \end{array}$$

p	q	$p \to q$	$\sim q$	$\sim p$
T	T	T	F	F
T	F	F	T	F
F	T	T	F	T
F	F	T	T	T

$\therefore$ The argument is valid.

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