Mathematics for Management

Chapter 2: Matrix Algebra

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Content:

- 2.1 Matrix Notation and Terminologies
- 2.2 Types of Matrices
- 2.3 Matrix Operations
- 2.4 Reduced Matrix
- 2.5 Determinant
- 2.6 Inverse Matrix

Expected Outcome:

Upon the completion of this course, students will have the ability to:

1. Apply the knowledge to solve the identified matrix algebra problems such as manipulate matrix algebra and determinants, apply row operations and elementary matrices.

Matrix Notation & Terminology

Definition (*m* x *n* **Matrix**)

a rectangular array of numbers enclosed within brackets that consisting of *m* horizontal row and *n* vertical columns,

$$\mathbf{A} = \left[a_{ij}\right]_{m\times n} = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ . & . & \dots & . \\ . & . & \dots & . \\ . & . & \dots & . \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{bmatrix}$$

where *ij* denotes the entry in the *i*th row and *j*th column.

A matrix usually denoted by **bold capital letters**. Eg:**A,B,C**

Type of Matrices

- <u>Row matrix</u>

  A matrix consisting of a single row.

  Example:

$$\begin{pmatrix} 3 & -7 & 10 \end{pmatrix}$$

- <u>Column matrix</u>

  A matrix consisting of a single column.

  Example:

$$\begin{pmatrix} 1 \\ 9 \\ 0 \end{pmatrix}$$

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- Square matrix

A matrix with the same number of rows and columns, $(m = n)$.

Example:

$$\begin{pmatrix} 4 & 1 \\ -6 & 2 \end{pmatrix}$$

- Null (zero) matrix, **O**

A matrix where all the elements are zero.

Example:

$$\begin{pmatrix} 0 & 0 & 0 \end{pmatrix}$$

- Identity matrix, **I**

A square matrix where the elements in the main diagonal are all 1's and the others are all zeros.

Example:

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Diagonal matrix

A square matrix where all its elements zeros, except for those in the main diagonal.

Note: Main diagonal is the entries that lie on the diagonal extending from upper left corner to the lower right corner.

Example:

$$\begin{pmatrix} 3 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 12 \end{pmatrix}$$

Symmetric matrix

A square matrix where the elements are symmetrical about the main diagonal.

$$A = A^T$$

Example:

$$\begin{pmatrix} 4 & -1 & 9 \\ -1 & 0 & 15 \\ 9 & 15 & 7 \end{pmatrix}$$

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➢ Upper triangular matrix

A square matrix where all the elements below the main diagonal are zeros.

Example:

$$\begin{pmatrix} -3 & 9 & -2 \\ 0 & 1 & 1 \\ 0 & 0 & 8 \end{pmatrix}$$

➢ Lower triangular matrix

A square matrix where all the elements above the main diagonal are zeros.

Example:

$$\begin{pmatrix} -3 & 0 & 0 \\ 4 & 1 & 0 \\ -7 & 5 & 8 \end{pmatrix}$$

Transpose of a Matrix

The transpose of m x n matrix **A**, denoted by $\mathbf{A}^T$, is the n x m matrix whose ith row is the ith column of **A**.
→ interchange its rows with its column.

$$\mathbf{A} = \begin{pmatrix} -3 & 3 & 6 \\ 4 & 1 & 0 \\ -7 & 5 & 8 \end{pmatrix}, \quad \mathbf{A}^T = \begin{pmatrix} -3 & 4 & -7 \\ 3 & 1 & 5 \\ 6 & 0 & 8 \end{pmatrix}$$

Properties of transpose matrix:

- $\left(\mathbf{A}^T\right)^T = \mathbf{A}$
- $\left(\mathbf{A}+\mathbf{B}\right)^T = \mathbf{A}^T + \mathbf{B}^T$

Example:

If $\mathbf{A} = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix}$, find $\mathbf{A}^{T}$.

Solution:

$$\mathbf{A}^{T} = \begin{pmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{pmatrix}$$

Exercises:

(a) If $\mathbf{S} = \begin{pmatrix} 0 \\ -3 \\ 7 \end{pmatrix}$, find $\mathbf{S}^{\mathrm{T}}$.

(b) If $B = \begin{pmatrix} 4 & 0 & 1 \\ 1 & 5 & 2 \\ -1 & 2 & 7 \end{pmatrix}$ find B^T

(c) If $C = \begin{pmatrix} 3 & 5 \end{pmatrix}$ find C^T

Equality of Matrices

Matrices $\mathbf{A} = \lfloor a_{ij} \rfloor$ and $\mathbf{B} = \lfloor b_{ij} \rfloor$ are equal if and only if they have the same size and $a_{ij} = b_{ij}$ for each *i* and *j* (corresponding entries are equal).

$$\mathbf{P} = \begin{pmatrix} 1 & 3 \\ 0 & 6 \end{pmatrix}, \quad \mathbf{Q} = \begin{pmatrix} 1 & 3 \\ 0 & 6 \end{pmatrix} \Rightarrow \text{ Matrices } \mathbf{P} \text{ and } \mathbf{Q} \text{ are equal}$$

$$\mathbf{P} = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix}, \quad \mathbf{Q} = \begin{pmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{pmatrix} \Rightarrow \text{ Matrices } \mathbf{P} \text{ and } \mathbf{Q} \text{ are not equal (different order)}$$

Example:

If $\begin{pmatrix} x & y+1 \\ 2z & 2 \end{pmatrix} = \begin{pmatrix} 2 & 7 \\ 4 & 2 \end{pmatrix}$, find the value of x, y and z.

Solution:

Comparing element a_{11};

$$x = 2$$

Comparing element a_{12};

$$y + 1 = 7$$
$$y = 7 - 1$$
$$y = 6$$

Comparing element a_{21};

$$2z = 4$$
$$z = \frac{4}{2}$$
$$z = 2$$

Exercises:

(a) If $\begin{pmatrix} 3 & p \\ q-4 & 0 \\ 10 & \frac{r}{5} \end{pmatrix} = \begin{pmatrix} 3 & -2 \\ 7 & 0 \\ 10 & 15 \end{pmatrix}$, find the value of p, q and r.

(b) If $\begin{pmatrix} 1/2 & p+q \\ 2r & r+p \end{pmatrix} = \begin{pmatrix} 1/2 & 2p \\ 4 & 8 \end{pmatrix}$, find the value of p, q and r.

Matrix Operations

1. Matrix Addition and Subtraction

If $\mathbf{A} = \lfloor a_{ij} \rfloor$ and $\mathbf{B} = \lfloor b_{ij} \rfloor$ are both *m* x *n* matrices, then the $\mathbf{A} \pm \mathbf{B}$ is the *m* x *n* matrix obtained by adding or subtracting corresponding entries of **A** and **B**, that is

$$\mathbf{A} \pm \mathbf{B} = \lfloor a_{ij} \pm b_{ij} \rfloor$$

If $\mathbf{A} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ and $\mathbf{B} = \begin{bmatrix} p & q \\ r & s \end{bmatrix}$

Hence, $\mathbf{A} \pm \mathbf{B} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \pm \begin{bmatrix} p & q \\ r & s \end{bmatrix} = \begin{bmatrix} a \pm p & b \pm q \\ c \pm r & d \pm s \end{bmatrix}$

Matrix Properties

If **A**, **B**, **C** and **O** have the same size,

(a) $\mathbf{A} \pm \mathbf{B} = \mathbf{B} \pm \mathbf{A}$ (commutative)

(b) $\mathbf{A} \pm (\mathbf{B} \pm \mathbf{C}) = (\mathbf{A} \pm \mathbf{B}) \pm \mathbf{C}$ (associative)

(c) $\mathbf{A} \pm \mathbf{O} = \mathbf{O} + \mathbf{A} = \mathbf{A}$ (identity)

Example:

Compute the following

a) $$\begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & -1 \\ 1 & 4 \end{bmatrix}$$

b) $$\begin{bmatrix} 1 & 0 \\ 3 & 4 \end{bmatrix} - \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

Solution:

(a) $\begin{pmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{pmatrix} + \begin{pmatrix} 1 & 0 \\ 0 & -1 \\ 1 & 4 \end{pmatrix} = \begin{pmatrix} 1+1 & 2+0 \\ 3+0 & 4+(-1) \\ 5+1 & 6+4 \end{pmatrix} = \begin{pmatrix} 2 & 2 \\ 3 & 3 \\ 6 & 10 \end{pmatrix}$

(b) $\begin{pmatrix} 1 & 0 \\ 3 & 4 \end{pmatrix} - \begin{pmatrix} 2 \\ 1 \end{pmatrix}$ cannot solve because both matrices have different order.

Exercise:

Given the matrices $\mathbf{P} = \begin{pmatrix} 4 & 2 & 6 \\ 1 & 3 & -7 \end{pmatrix}$, $\mathbf{Q} = \begin{pmatrix} 2 & 5 \\ 3 & 4 \end{pmatrix}$ and $\mathbf{R} = \begin{pmatrix} 3 & 9 & 0 \\ 4 & -1 & 8 \end{pmatrix}$. Find the following matrices, if exist.

(a) $\mathbf{P} + \mathbf{R}$
(b) $\mathbf{R} + \mathbf{Q}$
(c) $\mathbf{R} - \mathbf{P}$

2 Scalar Multiplication

If **A** is an *m* x *n* matrix and is a real number (also called a scalar), then by *k***A**, we denote the *m* x *n* matrix obtained by multiplying each entry in **A** by *k*, that is

$$\text{If } \mathbf{A} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

$$\text{then, } k\mathbf{A} = k\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} ka & kb \\ kc & kd \end{bmatrix}$$

where k is a real number

Properties

If matrices $\mathbf{A}$, $\mathbf{B}$ and $\mathbf{O}$ have the same order for any scalar k, k_1 and k_2,

- $k(\mathbf{A} \pm \mathbf{B}) = k\mathbf{A} \pm k\mathbf{B}$
- $(k_1 \pm k_2)\mathbf{A} = k_1\mathbf{A} \pm k_2\mathbf{A}$
- $k_1(k_2\mathbf{A}) = (k_1 k_2)\mathbf{A}$
- $0\mathbf{A} = \mathbf{O}$
- $k\mathbf{O} = \mathbf{O}$
- $(k\mathbf{A})^T = k\mathbf{A}^T$

Example:

Solve the equation $3\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} - \begin{pmatrix} -1 \\ 2 \end{pmatrix} = 4\begin{pmatrix} 3 \\ 1 \end{pmatrix}$.

Solution:

$$3\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} - \begin{pmatrix} -1 \\ 2 \end{pmatrix} = 4\begin{pmatrix} 3 \\ 1 \end{pmatrix}$$

$$\begin{pmatrix} 3x_1 \\ 3x_2 \end{pmatrix} - \begin{pmatrix} -1 \\ 2 \end{pmatrix} = \begin{pmatrix} 4\times 3 \\ 4\times 1 \end{pmatrix}$$

$$\begin{pmatrix} 3x_1 \\ 3x_2 \end{pmatrix} - \begin{pmatrix} -1 \\ 2 \end{pmatrix} = \begin{pmatrix} 12 \\ 4 \end{pmatrix}$$

$$3x_1 - (-1) = 12$$
$$3x_1 = 12 - 1$$
$$x_1 = \frac{11}{3}$$

$$3x_2 - 2 = 4$$
$$3x_2 = 4 + 2$$
$$x_2 = 2$$

Exercise:

Given the matrices $\mathbf{A}=\begin{pmatrix}3 & 2\\ 0 & 8\\ 6 & -2\end{pmatrix}$, $\mathbf{B}=\begin{pmatrix}9 & -1\\ 5 & -5\\ 3 & 4\end{pmatrix}$ and $\mathbf{C}=\begin{pmatrix}0 & -6\\ 9 & 2\end{pmatrix}$. Find the following matrices, if exist.

(a) $3\mathbf{A}+\mathbf{B}$

(b) $2\mathbf{B}-\mathbf{A}$

(c) $4\mathbf{C}+\mathbf{A}$

(d) $-\mathbf{A}+3\mathbf{B}$

3 Matrix Multiplication

Let **A** be an *m* x *n* matrix and **B** be an *n* x *p* matrix. Then the product **AB** is the *m* x *p* matrix **C** whose entry in row *i* and column *j* is obtained as follows: Sum the products formed by multiplying, in order, each entry (that is, first, second, etc.) in row *i* of **A** by the "corresponding" entry (that is, first, second, etc.) in column *j* of **B**.

A	$\times$	**B**	=	**C**
m x *n*		*n* x *p*		*m* x *p*

Properties

i. **A**(**BC**) = (**AB**)**C** (associative)

ii. **A**(**B** + **C**) = **AB** + **AC** or (**A**+**B**)**C** = **AC** + **BC** (distributive)

Example:

Compute the matrix product.

(a) $$\begin{pmatrix} 1 \\ 2 \end{pmatrix}\begin{pmatrix} 4 & 5 & 6 \\ 0 & -3 & 1 \end{pmatrix}$$

(b) $$\begin{pmatrix} 1 & 3 \\ 0 & -5 \\ 2 & 2 \end{pmatrix}\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$$

Solution:

(a) $\begin{pmatrix} 1 \\ 2 \end{pmatrix}_{2\times 1} \begin{pmatrix} 4 & 5 & 6 \\ 0 & -3 & 1 \end{pmatrix}_{2\times 3}$ cannot solve because did not satisfy $\underset{m\times \ell}{\mathbf{A}} . \underset{\ell\times n}{\mathbf{B}} = \underset{m\times n}{\mathbf{P}}$

(b) $$\begin{pmatrix} 1 & 3 \\ 0 & -5 \\ 2 & 2 \end{pmatrix}_{3\times 2} \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}_{2\times 2} = \begin{pmatrix} (1)(1)+(3)(3) & (1)(2)+(3)(4) \\ (0)(1)+(-5)(3) & (0)(2)+(-5)(4) \\ (2)(1)+(2)(3) & (2)(2)+(2)(4) \end{pmatrix}_{3\times 2}$$

$$= \begin{pmatrix} 1+9 & 2+12 \\ 0-15 & 0-20 \\ 2+6 & 4+8 \end{pmatrix}$$

$$= \begin{pmatrix} 10 & 14 \\ -15 & -20 \\ 8 & 12 \end{pmatrix}$$

Exercise:

Given the matrices $\mathbf{A} = \begin{pmatrix} 4 \\ 3 \end{pmatrix}$, $\mathbf{B} = \begin{pmatrix} 2 & 1 \\ -3 & 7 \end{pmatrix}$ and $\mathbf{C} = \begin{pmatrix} 3 & -2 & 1 \\ 2 & 4 & -6 \end{pmatrix}$. Find the following matrices, if exist.

(a) **AB**
(b) **BA**
(c) **CA**
(d) **BC**
(e) **CB**

Reduced Matrix

A matrix is said to be a **reduced matrix**, provided that all of the following are true:

- If **a row does not contain entirely of zeros**, then the first nonzero entry in the row, called the leading entry, **is 1**, whereas all other entries in the column in which the 1 appears are zeros.
- The first nonzero entry in each row is to the right of the first nonzero entry in each row above it.
- Any rows that consist entirely of zeros are at the bottom of the matrix

Example:

For each of the following matrices, determine whether it is reduced or not reduced.

1) $\begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$

2) $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$

3) $\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

4) $\begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$

5) $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

6) $\begin{bmatrix} 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

7) $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$

Solution:

(a) $\begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}$ no

(b) $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$ yes

(c) $\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ yes

(d) $\begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix}$ yes

(e) $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ no

(f) $\begin{pmatrix} 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 \end{pmatrix}$ no

(g) $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ no

Exercise:

For each of the following matrices, determine whether it is a reduced matrix or not.

(a) $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

(b) $\begin{pmatrix} 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 \end{pmatrix}$

(c) $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$

Elementary Row Operation

An augmented coefficient matrix is transformed into a row-equivalent matrix if any of the following row operation is performed.

i) two rows are interchanged: $R_i \leftrightarrow R_j$

ii) a row is multiplied by a non zero constant: $kR_i \rightarrow R_i$

iii) a constant multiple of one row is added to another row

$$kR_i + R_j \rightarrow R_j$$

$$R_i \leftrightarrow R_j$$

$$\left(\begin{array}{cc|c} 4 & 2 & 10 \\ 1 & 3 & 5 \end{array}\right) \quad R_1 \leftrightarrow R_2 \quad \left(\begin{array}{cc|c} 1 & 3 & 5 \\ 4 & 2 & 10 \end{array}\right)$$

$$kR_i \rightarrow R_i$$

$$\left(\begin{array}{cc|c} 1 & 3 & 5 \\ 4 & 2 & 10 \end{array}\right) \quad \frac{1}{4}R_2 \rightarrow R_2 \quad \left(\begin{array}{cc|c} 1 & 3 & 5 \\ 1 & 1/2 & 5/2 \end{array}\right)$$

$$kR_i + R_j \rightarrow R_j$$

$$\left(\begin{array}{cc|c} 1 & 3 & 5 \\ 1 & 1/2 & 5/2 \end{array}\right) \quad -R_1 + R_2 \rightarrow R_2 \quad \left(\begin{array}{cc|c} 1 & 3 & 5 \\ 0 & -5/2 & -5/2 \end{array}\right)$$

Types of Solutions

1) If the reduced augmented coefficient matrix has a row of the form

$$\left[\begin{array}{ccc|c} 0 & 0 & 0 & k \end{array}\right]$$

where k is a nonzero constant, then the linear system **AX** = **B** has no solution and inconsistent.

$$\left(\begin{array}{ccc|c} 1 & 0 & 3 & 7 \\ 0 & -1 & 2 & 4 \\ 0 & 0 & 0 & 2 \end{array}\right)$$

2) If the reduced augmented coefficient matrix has a last row of the form $\left[\begin{array}{ccc|c} 0 & j & k & l \end{array}\right]$

then the linear system **AX** = **B** has infinite number of solutions.

$$\left(\begin{array}{ccc|c} 1 & 0 & 0 & 7 \\ 0 & -1 & 2 & 4 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

3) If the reduced augmented coefficient matrix has a last row of the form $\left[\begin{array}{ccc|c} 0 & 0 & k & l \end{array}\right]$

then the linear system **AX** = **B** has unique solutions.

$$\left(\begin{array}{ccc|c} 1 & 0 & 0 & 3 \\ 0 & 1 & 0 & 4 \\ 0 & 0 & 1 & 8 \end{array}\right)$$

Solving System by Reduced Matrix

The objective is to show how to reduce a matrix and use matrix reduction to solve a linear system.

Let consider the system of linear equations:

$$3x + 5y = 25$$
$$x - 2y = 1$$

We can capture all the information contained in the system in the single augmented matrix as

$$\left(\begin{array}{cc|c} 3 & 5 & 25 \\ 1 & -2 & 1 \end{array}\right)$$

need a 1 here →
$$\left(\begin{array}{cc|c} 3 & 5 & 25 \\ 1 & -2 & 1 \end{array}\right) \quad R_1 \leftrightarrow R_2$$

need a 0 here →
$$\left(\begin{array}{cc|c} 1 & -2 & 1 \\ 3 & 5 & 25 \end{array}\right) \quad -3R_1 + R_2 \to R_2$$

$$\left(\begin{array}{cc|c} 1 & -2 & 1 \\ 0 & 11 & 22 \end{array}\right) \quad \frac{1}{11}R_2 \to R_2$$

need a 1 here ↗

$$\left(\begin{array}{cc|c} 1 & -2 & 1 \\ 0 & 1 & 2 \end{array}\right)$$

Example:

By using matrix reduction, solve the system of linear equations.

$$2x + 3y = -1$$
$$2x + y = 5$$
$$x + y = 1$$

Solution:

STEP 1 : Forming an augmented matrix

$$\left(\begin{array}{cc|c} 2 & 3 & -1 \\ 2 & 1 & 5 \\ 1 & 1 & 1 \end{array}\right)$$

STEP 2 : Matrix reduction using elementary row operations

$$\left(\begin{array}{cc|c} 2 & 3 & -1 \\ 2 & 1 & 5 \\ 1 & 1 & 1 \end{array}\right) \quad R_1 \leftrightarrow R_3$$

$$\left(\begin{array}{cc|c} 1 & 1 & 1 \\ 2 & 1 & 5 \\ 2 & 3 & -1 \end{array}\right) \quad -R_3 + R_2 \rightarrow R_2$$

$$\left(\begin{array}{cc|c} 1 & 1 & 1 \\ 0 & -2 & 6 \\ 2 & 3 & -1 \end{array}\right) \quad -2R_1 + R_3 \rightarrow R_3$$

$$\left(\begin{array}{cc|c} 1 & 1 & 1 \\ 0 & -2 & 6 \\ 0 & 1 & -3 \end{array}\right) \quad -\frac{R_2}{2} \rightarrow R_2$$

$$\left(\begin{array}{cc|c} 1 & 1 & 1 \\ 0 & 1 & -3 \\ 0 & 1 & -3 \end{array}\right) \quad -R_2 + R_3 \rightarrow R_3$$

$$\left(\begin{array}{cc|c} 1 & 1 & 1 \\ 0 & 1 & -3 \\ 0 & 0 & 0 \end{array}\right)$$

STEP 3 : Use back substitution to obtain all the answers

$$y = -3$$

$$\begin{aligned} x + y &= 1 \\ x - 3 &= 1 \\ x &= 4 \end{aligned}$$

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$$y = -3$$

$$\begin{aligned} x + y &= 1 \\ x - 3 &= 1 \\ x &= 4 \end{aligned}$$

STEP 4 : Summarizing up the solution to the system is,

$$x = 4 \quad , \quad y = -3$$

Exercise:

Solve the system of linear equations using matrix reduction.

$$x + 2y + 4z - 6 = 0$$
$$2z + y - 3 = 0$$
$$x + y + 2z - 1 = 0$$

Non-homogeneous and Homogeneous System

Definition:

The system

$$
\begin{aligned}
a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n &= c_1 \\
a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n &= c_2 \\
\vdots \quad\quad \vdots \quad\quad\quad\quad \vdots \quad\quad & \quad \vdots \\
a_{m1}x_1 + a_{m2}x_2 + \ldots + a_{mn}x_n &= c_m
\end{aligned}
$$

is called a **homogeneous system** if $c_1 = c_2 = \ldots = c_m = 0$.
The system is a **non-homogeneous system** if at least one of the c is not equal to zero.

THEOREM : Number of Solutions of a Homogeneous System

Let **A** be the reduced coefficient matrix of a homogeneous system of m linear equations in n unknowns. If **A** has exactly k nonzero rows, then $k \leq n$.

Morever

1) If $k < n$, the system has **infinitely many solutions**.
2) If $k = n$, the system has a **unique solution** (the trivial solution).

** In a homogeneous system, if the number of equations is less than the number of variables in the system, the system will always have infinitely many solutions.

Exercise:

Given that

$$\begin{aligned} x-2y+z&=0\\ 2x-y+5z&=0\\ x+y+4z&=0 \end{aligned}$$

Determine whether the above homogeneous system have a unique solution or infinitely many solutions. Then, solve the system.

Determinant

2x2 Matrix

Given $\mathbf{A} = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$, the determinant of $\mathbf{A}$ is:

$$\det \mathbf{A} = |\mathbf{A}| = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11}a_{22} - a_{21}a_{12}$$

Example:

Find the determinant for the following matrices.

(a) $\mathbf{A} = \begin{pmatrix} 4 & 2 \\ 5 & 3 \end{pmatrix}$

(b) $\mathbf{P} = \begin{pmatrix} -3 & 6 \\ 5 & -2 \end{pmatrix}$

Solution:

(a) $|\mathbf{A}| = \begin{vmatrix} 4 & 2 \\ 5 & 3 \end{vmatrix} = 4(3) - 5(2)$

$= 2$

(b) $|\mathbf{P}| = \begin{vmatrix} -3 & 6 \\ 5 & -2 \end{vmatrix} = -3(-2) - 5(6)$

$= -24$

Exercise:

Find the determinant for the following matrices.

(a) $\mathbf{A} = \begin{pmatrix} -4 & 1 \\ 2 & 8 \end{pmatrix}$

(b) $\mathbf{S} = \begin{pmatrix} 2 & 0 \\ 5 & -2 \end{pmatrix}$

3x3 Matrix

There are two methods to find determinant 3×3 matrices. Given $\mathbf{A}=\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$, the determinant of $\mathbf{A}$ is:

(1) Method 1

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = \left(a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32}\right) - \left(a_{13}a_{22}a_{31} + a_{11}a_{23}a_{32} + a_{12}a_{21}a_{33}\right)$$

(2) Method 2

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = a_{11}\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{12}\begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13}\begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$$

$$= a_{11}\left(a_{22}a_{33} - a_{23}a_{32}\right) - a_{12}\left(a_{21}a_{33} - a_{23}a_{31}\right) + a_{13}\left(a_{21}a_{32} - a_{22}a_{31}\right)$$

Example:

Find the determinant of $\begin{vmatrix} 2 & 3 & -1 \\ 4 & 0 & 5 \\ 3 & -2 & 1 \end{vmatrix}$

Method 2:

$$\begin{vmatrix} 2 & 3 & -1 \\ 4 & 0 & 5 \\ 3 & -2 & 1 \end{vmatrix} = 2\begin{vmatrix} 0 & 5 \\ -2 & 1 \end{vmatrix} - 3\begin{vmatrix} 4 & 5 \\ 3 & 1 \end{vmatrix} + (-1)\begin{vmatrix} 4 & 0 \\ 3 & -2 \end{vmatrix}$$

$$= 2(0+10) - 3(4-15) - (-8-0)$$

$$= 20 + 33 + 8$$

$$= 61$$

Exercise:

Find the determinant of $\begin{vmatrix} 4 & -1 & 6 \\ 1 & 9 & 3 \\ 2 & 1 & 4 \end{vmatrix}$

Inverse

- Objective: To determine the inverse of an invertible matrix and to use inverses to solve system.

- Definition: If **A** is a square matrix and there exists a matrix **C** such that **CA**=**I**, then **C** is called an inverse of **A**, and **A** is said to be invertible.

The matrix $\mathbf{A}$ is invertible if there exists a matrix $\mathbf{A}^{-1}$ such that

$$\mathbf{A}^{-1}\mathbf{A} = \mathbf{A}\mathbf{A}^{-1} = \mathbf{I}$$

2.6.1) Inverse by Adjoint Method

If **A** is a square matrix, the transpose matrix of matrix cofactor **A** is known as adjoint of matrix **A** and is denoted by $Adj(\mathbf{A}) = \mathbf{C}^{\mathrm{T}}$, where **C** is the cofactor of matrix **A**. If $|\mathbf{A}| \neq 0$ (non singular), then the inverse matrix

$$\mathbf{A}^{-1} = \frac{1}{|\mathbf{A}|} Adj(\mathbf{A})$$

2x2 Matrix

In the case of a 2x2 matrix, a simple formula exists to find its adjoint matrix

If $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ then $Adj(A) = \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$

Example:

Find the adjoint matrix of $\begin{pmatrix} -9 & 0 \\ -1 & 3 \end{pmatrix}$.

Solution:

$$\begin{pmatrix} 3 & 0 \\ 1 & -9 \end{pmatrix}$$

Example:

Given $\mathbf{A} = \begin{pmatrix} 3 & 1 \\ 4 & 2 \end{pmatrix}$, find its inverse.

Solution:

$$\mathbf{A}^{-1} = \frac{1}{(3)(2)-(4)(1)}\begin{pmatrix} 2 & -1 \\ -4 & 3 \end{pmatrix}$$

$$= \frac{1}{2}\begin{pmatrix} 2 & -1 \\ -4 & 3 \end{pmatrix}$$

$$= \begin{pmatrix} \frac{2}{2} & -\frac{1}{2} \\ -\frac{4}{2} & \frac{3}{2} \end{pmatrix}$$

$$= \begin{pmatrix} 1 & -\frac{1}{2} \\ -2 & \frac{3}{2} \end{pmatrix}$$

3x3 Matrix

- Consider the matrix **A** of order 3 x 3, $\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$

  Suppose that we choose any entry, say a_{21}, and strike out the row and column that pass through a_{21}, then we will obtain

$$M_{21} = \begin{vmatrix} a_{12} & a_{13} \\ a_{32} & a_{33} \end{vmatrix} = a_{12}a_{33} - a_{13}a_{32}$$

which called the **minor** of the entry

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The value of $C_{ij} = (-1)^{i+j} M_{ij}$ is called the **cofactor** of the entry as summarize in the diagram below.

$$\begin{vmatrix} + & - & + \\ - & + & - \\ + & - & + \end{vmatrix}$$

So, we can get the adjoint matrix,

$$adj\,\mathbf{A} = \mathbf{C}^{\mathrm{T}} = \begin{pmatrix} C_{11} & C_{12} & C_{13} \\ C_{21} & C_{22} & C_{23} \\ C_{31} & C_{32} & C_{33} \end{pmatrix}^{\mathrm{T}}$$

Example:

Given $\mathbf{A} = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$, find

(a) M_{11}

(b) M_{13}

(c) M_{22}

(d) C_{32}

Solution:

(a) $\mathbf{A}=\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} \rightarrow M_{11}=\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix}$

(b) $\mathbf{A}=\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} \rightarrow M_{13}=\begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$

(c) $\mathbf{A}=\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} \rightarrow M_{22}=\begin{vmatrix} a_{11} & a_{13} \\ a_{31} & a_{33} \end{vmatrix}$

(d) $\mathbf{A}=\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} \rightarrow C_{32}=-M_{32}=-\begin{vmatrix} a_{11} & a_{13} \\ a_{21} & a_{23} \end{vmatrix}$

Example:

Given $\mathbf{A} = \begin{pmatrix} 1 & -2 & 0 \\ 3 & 1 & 5 \\ -1 & 2 & 3 \end{pmatrix}$, find $adj\ \mathbf{A}$. Hence, find its inverse, $\mathbf{A}^{-1}$.

Solution:

$$adj\,\mathbf{A} = \begin{pmatrix} +\begin{vmatrix} 1 & 5 \\ 2 & 3 \end{vmatrix} & -\begin{vmatrix} 3 & 5 \\ -1 & 3 \end{vmatrix} & +\begin{vmatrix} 3 & 1 \\ -1 & 2 \end{vmatrix} \\ -\begin{vmatrix} -2 & 0 \\ 2 & 3 \end{vmatrix} & +\begin{vmatrix} 1 & 0 \\ -1 & 3 \end{vmatrix} & -\begin{vmatrix} 1 & -2 \\ -1 & 2 \end{vmatrix} \\ +\begin{vmatrix} -2 & 0 \\ 1 & 5 \end{vmatrix} & -\begin{vmatrix} 1 & 0 \\ 3 & 5 \end{vmatrix} & +\begin{vmatrix} 1 & -2 \\ 3 & 1 \end{vmatrix} \end{pmatrix}^T$$

$$= \begin{pmatrix} -7 & -14 & 7 \\ 6 & 3 & 0 \\ -10 & -5 & 7 \end{pmatrix}^T$$

$$= \begin{pmatrix} -7 & 6 & -10 \\ -14 & 3 & -5 \\ 7 & 0 & 7 \end{pmatrix}$$

$$|\mathbf{A}| = \begin{vmatrix} 1 & 5 \\ 2 & 3 \end{vmatrix} - (-2)\begin{vmatrix} 3 & 5 \\ -1 & 3 \end{vmatrix} + 0$$

$$= -7 + 28$$

$$= 21$$

$$\mathbf{A}^{-1} = \frac{1}{21}\begin{pmatrix} -7 & 6 & -10 \\ -14 & 3 & -5 \\ 7 & 0 & 7 \end{pmatrix}$$

$$= \begin{pmatrix} -\frac{1}{3} & \frac{6}{21} & -\frac{10}{21} \\ -\frac{2}{3} & \frac{3}{21} & -\frac{5}{21} \\ \frac{1}{3} & 0 & \frac{1}{3} \end{pmatrix}$$

Exercises:

1. Given $\mathbf{A} = \begin{pmatrix} 5 & 2 & -1 \\ 7 & 1 & 0 \\ 3 & -1 & 2 \end{pmatrix}$, find

(a) M_{12}

(b) M_{23}

(c) M_{33}

(d) C_{21}

2. Given $\mathbf{A} = \begin{pmatrix} 1 & -2 & 0 \\ 3 & 1 & 5 \\ -1 & 2 & 3 \end{pmatrix}$, find $adj\ \mathbf{A}$. Hence, find its inverse, $\mathbf{A}^{-1}$.

Inverse Matrix by Elementary Row Operation

If $\left[\mathbf{A}\middle|\mathbf{I}\right]$ can be transformed by elementary row operation to $\left[\mathbf{I}\middle|A^{-1}\right]$, then the resulting matrix **M** is .

If a matrix **A** does not reduce to **I**, then $\mathbf{A}^{-1}$ does not exist.

Example:

Example 2.6.2.1:

Find the inverse of $P = \begin{pmatrix} 3 & 3 & 4 \\ 4 & 3 & 2 \\ 2 & 2 & 3 \end{pmatrix}$ using elementary row operations.

Solution:

$$\left(\begin{array}{ccc|ccc} 3 & 3 & 4 & 1 & 0 & 0 \\ 4 & 3 & 2 & 0 & 1 & 0 \\ 2 & 2 & 3 & 0 & 0 & 1 \end{array}\right) \quad \frac{1}{3}R_1 \to R_1$$

$$\left(\begin{array}{ccc|ccc} 1 & 1 & \frac{4}{3} & \frac{1}{3} & 0 & 0 \\ 4 & 3 & 2 & 0 & 1 & 0 \\ 2 & 2 & 3 & 0 & 0 & 1 \end{array}\right) \quad -2R_3 + R_2 \to R_2$$

$$\left(\begin{array}{ccc|ccc} 1 & 1 & \frac{4}{3} & \frac{1}{3} & 0 & 0 \\ 0 & -1 & -4 & 0 & 1 & -2 \\ 2 & 2 & 3 & 0 & 0 & 1 \end{array}\right) \quad -2R_1 + R_3 \to R_3$$

$$\left(\begin{array}{ccc|ccc} 1 & 1 & \frac{4}{3} & \frac{1}{3} & 0 & 0 \\ 0 & -1 & -4 & 0 & 1 & -2 \\ 0 & 0 & \frac{1}{3} & -\frac{2}{3} & 0 & 1 \end{array}\right) \quad R_2 + R_1 \to R_1$$

$$\left(\begin{array}{ccc|ccc} 1 & 0 & -\frac{8}{3} & \frac{1}{3} & 1 & -2 \\ 0 & -1 & -4 & 0 & 1 & -2 \\ 0 & 0 & \frac{1}{3} & -\frac{2}{3} & 0 & 1 \end{array}\right) \quad -R_2 \to R_2$$

$$\left(\begin{array}{ccc|ccc} 1 & 0 & -\frac{8}{3} & \frac{1}{3} & 1 & -2 \\ 0 & 1 & 4 & 0 & -1 & 2 \\ 0 & 0 & \frac{1}{3} & -\frac{2}{3} & 0 & 1 \end{array}\right) \quad 8R_3 + R_1 \to R_1$$

$$\left(\begin{array}{ccc|ccc} 1 & 0 & 0 & -5 & 1 & 6 \\ 0 & 1 & 4 & 0 & -1 & 2 \\ 0 & 0 & \frac{1}{3} & -\frac{2}{3} & 0 & 1 \end{array}\right) \quad -12R_3 + R_2 \to R_2$$

$$\left(\begin{array}{ccc|ccc} 1 & 0 & 0 & -5 & 1 & 6 \\ 0 & 1 & 0 & 8 & -1 & -10 \\ 0 & 0 & \frac{1}{3} & -\frac{2}{3} & 0 & 1 \end{array}\right) \quad 3R_3 \to R_3$$

$$\left(\begin{array}{ccc|ccc} 1 & 0 & 0 & -5 & 1 & 6 \\ 0 & 1 & 0 & 8 & -1 & -10 \\ 0 & 0 & 1 & -2 & 0 & 3 \end{array}\right)$$

$$\therefore\ \mathbf{P}^{-1} = \begin{pmatrix} -5 & 1 & 6 \\ 8 & -1 & -10 \\ -2 & 0 & 3 \end{pmatrix}$$

Exercise:

Find the inverse of $\mathbf{A} = \begin{pmatrix} 3 & 1 \\ 4 & 2 \end{pmatrix}$ using elementary row operations.

Solving System by Inverse Matrix

Solve the system

$$\begin{aligned} x_1 \qquad\quad - \ 2x_3 &= \ \ 1 \\ 4x_1 - 2x_2 + \quad x_3 &= \ \ 2 \\ x_1 + 2x_2 - 10x_3 &= -1 \end{aligned}$$

by finding the inverse of the coefficient matrix.

Solve by using adjoint matrix

$$\begin{pmatrix} 1 & 0 & -2 \\ 4 & -2 & 1 \\ 1 & 2 & -10 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix}$$

$$adj\ \mathbf{A} = \begin{pmatrix} +\begin{vmatrix} -2 & 1 \\ 2 & -10 \end{vmatrix} & -\begin{vmatrix} 4 & 1 \\ 1 & -10 \end{vmatrix} & +\begin{vmatrix} 4 & -2 \\ 1 & 2 \end{vmatrix} \\ -\begin{vmatrix} 0 & -2 \\ 2 & -10 \end{vmatrix} & +\begin{vmatrix} 1 & -2 \\ 1 & -10 \end{vmatrix} & -\begin{vmatrix} 1 & 0 \\ 1 & 2 \end{vmatrix} \\ +\begin{vmatrix} 0 & -2 \\ -2 & 1 \end{vmatrix} & -\begin{vmatrix} 1 & -2 \\ 4 & 1 \end{vmatrix} & +\begin{vmatrix} 1 & 0 \\ 4 & -2 \end{vmatrix} \end{pmatrix}^T$$

$$= \begin{pmatrix} 18 & 41 & 10 \\ -4 & -8 & -2 \\ -4 & -9 & -2 \end{pmatrix}^T$$

$$= \begin{pmatrix} 18 & -4 & -4 \\ 41 & -8 & -9 \\ 10 & -2 & -2 \end{pmatrix}$$

$$|\mathbf{A}| = 1\begin{vmatrix} -2 & 1 \\ 2 & -10 \end{vmatrix} - 0 + (-2)\begin{vmatrix} 4 & -2 \\ 1 & 2 \end{vmatrix}$$

$$= 18 - 20$$

$$= -2$$

$$\mathbf{A}^{-1} = -\frac{1}{2}\begin{pmatrix} 18 & -4 & -4 \\ 41 & -8 & -9 \\ 10 & -2 & -2 \end{pmatrix}$$

$$= \begin{pmatrix} -9 & 2 & 2 \\ -\frac{41}{2} & 4 & \frac{9}{2} \\ -5 & 1 & 1 \end{pmatrix}$$

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} -9 & 2 & 2 \\ -\frac{41}{2} & 4 & \frac{9}{2} \\ -5 & 1 & 1 \end{pmatrix}\begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix} = \begin{pmatrix} -7 \\ -17 \\ -4 \end{pmatrix}$$

Solve by using ERO

$$\left(\begin{array}{ccc|ccc} 1 & 0 & -2 & 1 & 0 & 0 \\ 4 & -2 & 1 & 0 & 1 & 0 \\ 1 & 2 & -10 & 0 & 0 & 1 \end{array}\right) \quad -4R_1 + R_2 \rightarrow R_2$$

$$\left(\begin{array}{ccc|ccc} 1 & 0 & -2 & 1 & 0 & 0 \\ 0 & -2 & 9 & -4 & 1 & 0 \\ 1 & 2 & -10 & 0 & 0 & 1 \end{array}\right) \quad -R_1 + R_3 \rightarrow R_3$$

$$\left(\begin{array}{ccc|ccc} 1 & 0 & -2 & 1 & 0 & 0 \\ 0 & -2 & 9 & -4 & 1 & 0 \\ 0 & 2 & -8 & -1 & 0 & 1 \end{array}\right) \quad -\frac{1}{2}R_2 \rightarrow R_2$$

$$\left(\begin{array}{ccc|ccc} 1 & 0 & -2 & 1 & 0 & 0 \\ 0 & 1 & -\frac{9}{2} & 2 & -\frac{1}{2} & 0 \\ 0 & 2 & -8 & -1 & 0 & 1 \end{array}\right) \quad -2R_2 + R_3 \rightarrow R_3$$

$$\left(\begin{array}{ccc|ccc} 1 & 0 & -2 & 1 & 0 & 0 \\ 0 & 1 & -\frac{9}{2} & 2 & -\frac{1}{2} & 0 \\ 0 & 0 & 1 & -5 & 1 & 1 \end{array}\right) \quad 2R_3 + R_1 \rightarrow R_1$$

$$\left(\begin{array}{ccc|ccc} 1 & 0 & 0 & -9 & 2 & 2 \\ 0 & 1 & -\frac{9}{2} & 2 & -\frac{1}{2} & 0 \\ 0 & 0 & 1 & -5 & 1 & 1 \end{array}\right) \quad \frac{9}{2}R_3 + R_2 \rightarrow R_2$$

$$\left(\begin{array}{ccc|ccc} 1 & 0 & 0 & -9 & 2 & 2 \\ 0 & 1 & 0 & -\frac{41}{2} & 4 & \frac{9}{2} \\ 0 & 0 & 1 & -5 & 1 & 1 \end{array}\right)$$

$$\mathbf{A}^{-1} = \begin{pmatrix} -9 & 2 & 2 \\ -\frac{41}{2} & 4 & \frac{9}{2} \\ -5 & 1 & 1 \end{pmatrix}$$

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} -9 & 2 & 2 \\ -\frac{41}{2} & 4 & \frac{9}{2} \\ -5 & 1 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix} = \begin{pmatrix} -7 \\ -17 \\ -4 \end{pmatrix}$$

Exercise:

Solve the system of linear equation by finding the inverse of the coefficient matrix.

$$x + 2y + z = 12$$
$$3y - 4z = 4$$
$$2x + y + z = 10$$

THE END

~THANK YOU~

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