Mathematics for Management

Chapter 1: Functions

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Content:

- 1.0 Introduction
- 1.1 Functions
- 1.2 Special Functions
- 1.3 Combination of Functions
- 1.4 Composition of Functions
- 1.5 Inverse Functions

Expected Outcome:

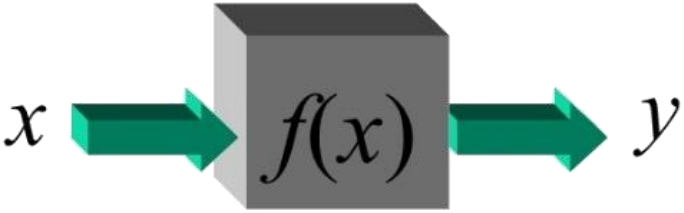
Upon the completion of this course, students will have the ability to:

1. Obtain the domain, function values, equality of function and difference quotient of a function.
2. Identify the types of special functions i.e. constant functions, polynomial functions, rational functions, case-defined function, and absolute value function.
3. Find the solution of combination of functions, composition of functions and inverse functions.

1.0 Introduction

Definition:

- A function is a relationship in which each input number is paired with *exactly one* output number.



- The set of all input numbers is called the domain of the function (independent variable)
- The set of all output numbers is called the range (dependent variable)

1.1 Functions

$$y = f(x)$$

Output — y

Name of Function — f

Input — x

Domain

Domain is the set of all input numbers

Three different cases of domain:

1

$$f(x) = \frac{x}{x^2 + x - 6}$$

$$x^2 + x - 6 \neq 0$$

$$(x-2)(x+3) \neq 0$$

$$x \neq 2, -3$$

Fractional functions
(The denominator should not be equal to zero)

Domain: All real numbers except 2 and -1

Domain

2

$$g(k)=\sqrt{2k-5}$$

$$2k-5\geq 0$$

$$2t\geq 5$$

$$t\geq \frac{5}{2}$$

Domain $\left[\frac{5}{2},\infty\right)$

Square root functions
(Under the square root, the all numbers should be in positive)

3

$$g(x)=6x^2-x+7$$

Polynomial functions
(all numbers can be as an input)

Domain: All real numbers

Function Values

Function values is the output of the function corresponding to the input

Example 1.1

Let $g(p) = 3p^2 - p + 5$ find $g(x), g(c^2), g(k+h)$

Solution:

$$g(z) = 3z^2 - z + 5$$

$$g(r^2) = 3(r^2)^2 - r^2 + 5 = 3r^4 - r^2 + 5$$

$$\begin{aligned} g(x+h) &= 3(x+h)^2 - (x+h) + 5 \\ &= 3(x^2 + 2hx + h^2) - x - h + 5 \\ &= 3x^2 + 6hx + 3h^2 - x - h + 5 \end{aligned}$$

Exercises

Function values is the output of the function corresponding to the input

Example 1.1

Let $g(z) = 3z^2 - z + 5$ find $g(x), g(h^2), g(k+h)$

Solution: $g(x) = 3x^2 - x + 5$

$$g(h^2) = 3(h^2)^2 - h^2 + 5 = 3h^4 - h^2 + 5$$

$$g(k+h) = 3(k+h)^2 - (k+h) + 5$$

$$= 3(k^2 + 2hk + h^2) - k - h + 5$$

$$= 3k^2 + 6hk + 3h^2 - k - h + 5$$

Exercises:

Find the **domain** and the **function values** for the following functions

(a) $f(x)=\sqrt{x+8}$; $f(3), f(8), f(-4)$

(b) $g(x)=x^3+x^2$; $g(2), g(-1), g(t)$

(c) $h(x)=\dfrac{1}{x-6}$

Equality of Functions

To say that two functions f and g are equal, denoted $f = g$, is to say that

- the domain of f = domain of g
- for every x in the domain of f and g, output $\boldsymbol{f(x)}$ = output $\boldsymbol{g(x)}$

Example:

Determine whether the following functions are equal.

(a) $f(x) = (x+1)^2$ and $g(x) = x^2 + 2x + 1$

(b) $f(x) = x^2$ and $g(x) = x^2$ for $x \geq 0$

SOLUTION:

(a) Domain $f(x)$ = all real numbers.
Domain $g(x)$ = all real numbers.

Therefore, Domain $f(x)$ = Domain $g(x)$

$$f(x) = (x+1)^2$$
$$= x^2 + 2x + 1$$

$$g(x) = x^2 + 2x + 1$$

$$\therefore f(x) = g(x)$$

Therefore, these two functions are equal.

(b) If we have $f(x) = x^2$, with no explicit mention of domain, and $g(x) = x^2$ for $x \geq 0$, then domain $f(x) \neq g(x)$. Here the domain of f is all real numbers and the domain for g is $[0, \infty)$. Therefore, these two functions are not equal.

Exercises:

Determine whether the following functions are **equal**.

(a) $f(x)=(x+1)^2$ and $g(x)=x+2$

(b) $g(x)=x+2$ and $k(x)=\begin{cases} x+2 & \text{if } x\neq 1 \\ 3 & \text{if } x=1 \end{cases}$

DIFFERENT QUOTIENT

The difference quotient of a function is an important mathematical concept. The expression is

$$\frac{f(x+h)-f(x)}{h}$$

Example: If $f(x)=x^2$ find $\dfrac{f(x+h)-f(x)}{h}$

$$\frac{f(x+h)-f(x)}{h}=\frac{(x+h)^2-x^2}{h}$$

$$=\frac{x^2+2hx+h^2-x^2}{h}$$

$$=\frac{2hx+h^2}{h}$$

$$=\frac{h(2x+h)}{h}$$

$$=2x+h$$

Different Quotient

The difference quotient of a function is an important mathematical concept. The expression is

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Example: If $f(x)=x^2$ find $\dfrac{f(x+h)-f(x)}{h}$

$$\frac{f(x+h)-f(x)}{h}=\frac{(x+h)^2-x^2}{h}$$

$$=\frac{x^2+2hx+h^2-x^2}{h}$$

$$=\frac{2hx+h^2}{h}=\frac{h(2x+h)}{h}=2x+h$$

Exercises:

Find the **difference quotient** of the following functions.

(a) Find $\frac{-f(x+h)+f(x)}{h}$ if $f(x)=\frac{x}{2}+1$

(b) If $f(x)=4x-5$, find $\frac{f(x+2h)-f(x)}{4h}$.

Special Functions

- Constant Functions
- Polynomial Functions
- Rational Functions
- Case-Defined Function
- Absolute-Value Function

Constant Functions

- A function of the form $h(x)=c$, where c is a constant, is called a constant function.

Let $h(x)=7$, so

$$h(12)=7,$$
$$h(-385)=7,$$
$$h(x+5)=7$$

We call h a constant function because all the function value are the same.

Polynomial Functions

- In general, a function of the form

$$f(x) = c_n x^n + c_{n-1} x^{n-1} + \ldots + c_1 x + c_0$$

Where n is a nonnegative integer and $c_n, c_{n-1}, \ldots, c_0$ are constant with $c_n \neq 0$, is called a polynomial function

Example:

(a) $f(x) = x^3 - 6x^2 + 7 \rightarrow$ is a polynomial function of degree 3 with leading coefficient 1.

(b) $g(x) = \dfrac{2x}{3} \rightarrow$ is a linear function with leading coefficient $\dfrac{2}{3}$.

(c) $f(x) = \dfrac{2}{x^3} \rightarrow$ is NOT a polynomial function. Because $f(x) = x^{-3}$ and the exponent for x is not a nonnegative integer, this function does not have the proper form for a polynomial.

Rational Functions

- A function that is a quotient of polynomial functions is called a rational function.

(a) $f(x)=\dfrac{x^2-6x}{x+5}$ → is a rational function, since the numerator and denominator are each polynomials. Note that this rational function is not defined for $x=5$.

(b) $g(x)=2x+3$ → is a rational function, since $2x+3=\dfrac{2x+3}{1}$. In fact, every polynomial function is also a rational function.

Case-Defined Function

Let

$$F(s)=\begin{cases}1 & if \quad -3\le s\le 2\\ 0 & if \quad 2\le s\le 3\\ s-3 & if \quad 3< s\le 8\end{cases}$$

This is called a case-defined function because the rule for specifying it is given by rules for each of several disjoint cases.

Absolute-Value Function

- The function $|-|(x) = |x|$ is called the absolute- value function. $|x|$ is defined by

$$|x| = \begin{cases} x & if \;\; x \geq 0 \\ -x & if \;\; x < 0 \end{cases}$$

Exercises:

Classify each of the special function. Then, find the function values.

(a) $f(x)=8;\ f(2), f(t+8), f(-\sqrt{17})$

(b) $H(x)=\dfrac{1}{\pi}-3x^5+2x^6+x^7;\ H(2), H(-2), H(1)$

(c) $h(x)=\dfrac{x^2+x}{x^3+4};\ h(2), h(t+8), h(0)$

(d) $G(\theta)=\begin{cases}2\theta-5 & \text{if } \theta\le 2\\ \theta^2-3\theta+1 & \text{if } \theta>2\end{cases}$

$G(3), G(-3), G(2)$

(e) $g(x)=|x-3|;\ g(10), g(3), g(-3)$

Combinations of Functions

- There are several ways of combining two functions to create a new function. For example, we can combine the functions by addition, subtraction, multiplication, division, multiplication by a constant, and composition.

- Suppose f and g are the functions given by

$$f(x) = x^3 \text{ and } \quad g(x) = 5x$$

Adding $f(x)$ and $g(x)$ gives

$$f(x) + g(x) = x^3 + 5x$$

- In general, for any function *f* and *g*, we define the sum *f+g*, the difference *f-g*, the product *fg* and the quotient *f/g* and scalar product *cf(x)* as follows:

$$i. (f+g)(x) = f(x) + g(x)$$

$$ii. (f-g)(x) = f(x) - g(x)$$

$$iii. (fg)(x) = f(x).g(x)$$

$$iv. \frac{f}{g}(x) = \frac{f(x)}{g(x)}, g(x) \neq 0$$

$$v. (cf)(x) = c.f(x)$$

Example:

If $f(x)=x^2$ and $g(x)=3x$, therefore we have

Solution:

(a) $(f+g)(x)=f(x)+g(x)=x^2+3x$

(b) $(f-g)(x)=f(x)-g(x)=x^2-3x$

(c) $(fg)(x)=f(x)\cdot g(x)=x^2(3x)=3x^3$

(d) $\dfrac{f}{g}(x)=\dfrac{f(x)}{g(x)}=\dfrac{x^2}{3x}=\dfrac{x}{3}$ for $x\neq 0$

(e) $(\sqrt{2}f)(x)=\sqrt{2}f(x)=\sqrt{2}x^2$

Exercises:

If $f(x) = 3x - 1$ and $g(x) = x^2 + 3x$, find

(a) $(f + g)(3)$

(b) $(f - g)(\frac{1}{2})$

(c) $(fg)(-\frac{1}{2})$

(d) $\frac{f}{g}(-2)$

(e) $(3f)(-\sqrt{2})$

Composition of Functions

Definition:

If f and g are functions, the composite of f with g is the function $f \circ g$ define by

$$(f \circ g)(x) = f(g(x))$$

where the domain of $f \circ g$ is the set of all those x in the domain of g such that $g(x)$ is in the domain of f

Example:

If $f(x)=\sqrt{x}$ and $g(x)=x+1$, find

(a) $(f\circ g)(x)$, then $(f\circ g)(8)$.

(b) $(g\circ f)(x)$, then $(g\circ f)\left(\frac{1}{4}\right)$

Solution:

(a) $$(f \circ g)(x) = f(g(x))$$
$$= f(x+1)$$
$$= \sqrt{x+1}$$

The domain is all $x \geq -1$, or equivalently, the interval $[-1, \infty)$.

When $x = 8$, then $(f \circ g)(8) = \sqrt{8+1} = \sqrt{9} = 3$.

(b) $$(g \circ f)(x) = g(f(x))$$
$$= g(\sqrt{x})$$
$$= \sqrt{x} + 1$$

The domain is all $x \geq 0$, or equivalently, the interval $[0, \infty)$.

When $x = \frac{1}{4}$, then $(g \circ f)\left(\frac{1}{4}\right) = \sqrt{\frac{1}{4}} + 1 = \frac{1}{\sqrt{4}} + 1 = \frac{1}{2} + 1 = \frac{3}{2}$.

Exercises:

If $f(x)=2x+1$ and $g(x)=\dfrac{3}{2x+1}$, find

(a) $(g\circ f)(-2)$

(b) $(f\circ g)\left(\dfrac{1}{2}\right)$

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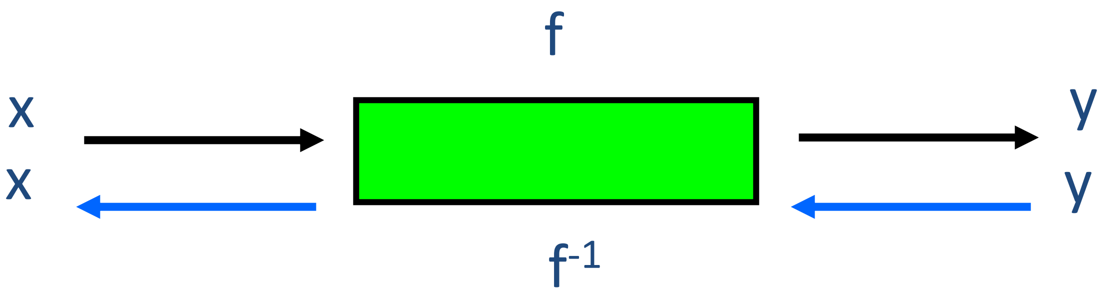
Inverse Functions

- Only exist for one-to-one function



$$f(x) = y \rightarrow f^{-1}(y) = x$$

- **Remarks:** To verify that is f^{-1} the inverse of f, show that

$$f^{-1}[f(x)] = f[f^{-1}(x)] = x$$

A function f that satisfies

for all a and b, if $f(a) = f(b)$, then $a = b$

is called **one-to-one** function.

3 steps to find the inverse of a function, f:

STEP 1: Replace $f(x)$ with y

STEP 2: Solve for x in term of y obtaining $x = g(y)$

STEP 3: Replace x with $f^{-1}(x)$. Then, $f^{-1}(x) = g(x)$

Example:

If $f(x) = (x-1)^2$ for $x \geq 1$, find $f^{-1}(x)$.

Solution:

Let $y = (x-1)^2$, for $x \geq 1$.

Then, $x - 1 = \sqrt{y}$ and hence $x = \sqrt{y} + 1$.

Therefore, we have $f^{-1}(x) = \sqrt{x} + 1$.

Check:

$$f^{-1}\left(f(x)\right) = f\left((x-1)^2\right) = \sqrt{(x-1)^2} + 1$$

$$= x - 1 + 1 \quad = x$$

$$f\left(f^{-1}(x)\right) = f\left(\sqrt{x} + 1\right) = \left[\left(\sqrt{x} + 1\right) - 1\right]^2$$

$$= \left(\sqrt{x} + 1\right)^2 - 2\left(\sqrt{x} + 1\right) + 1 \quad = x$$

Therefore, the inverse of function f given above is $f^{-1}(x) = \sqrt{x} + 1$.

Exercises:

Find the inverse of the given function.

(a) $f(x) = \dfrac{3x-1}{x+5}$

(b) $g(x) = (2x+8)^3$

THE END

~THANK YOU~

Author Information

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