DYNAMICS

Planar Kinematics of a Rigid Body (Relative Motion using Rotating Axes)

by:

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Relative Motion using Rotating Axes

- Aims
  - To revise on the relative motion with translating axis.
  - To introduce the relative motion using rotating axes.

- Expected Outcomes
  - Students are able to distinguish between the case of translating and rotating axes.
  - Students are able to determine velocity and acceleration in the case of relative motion using rotating axes.

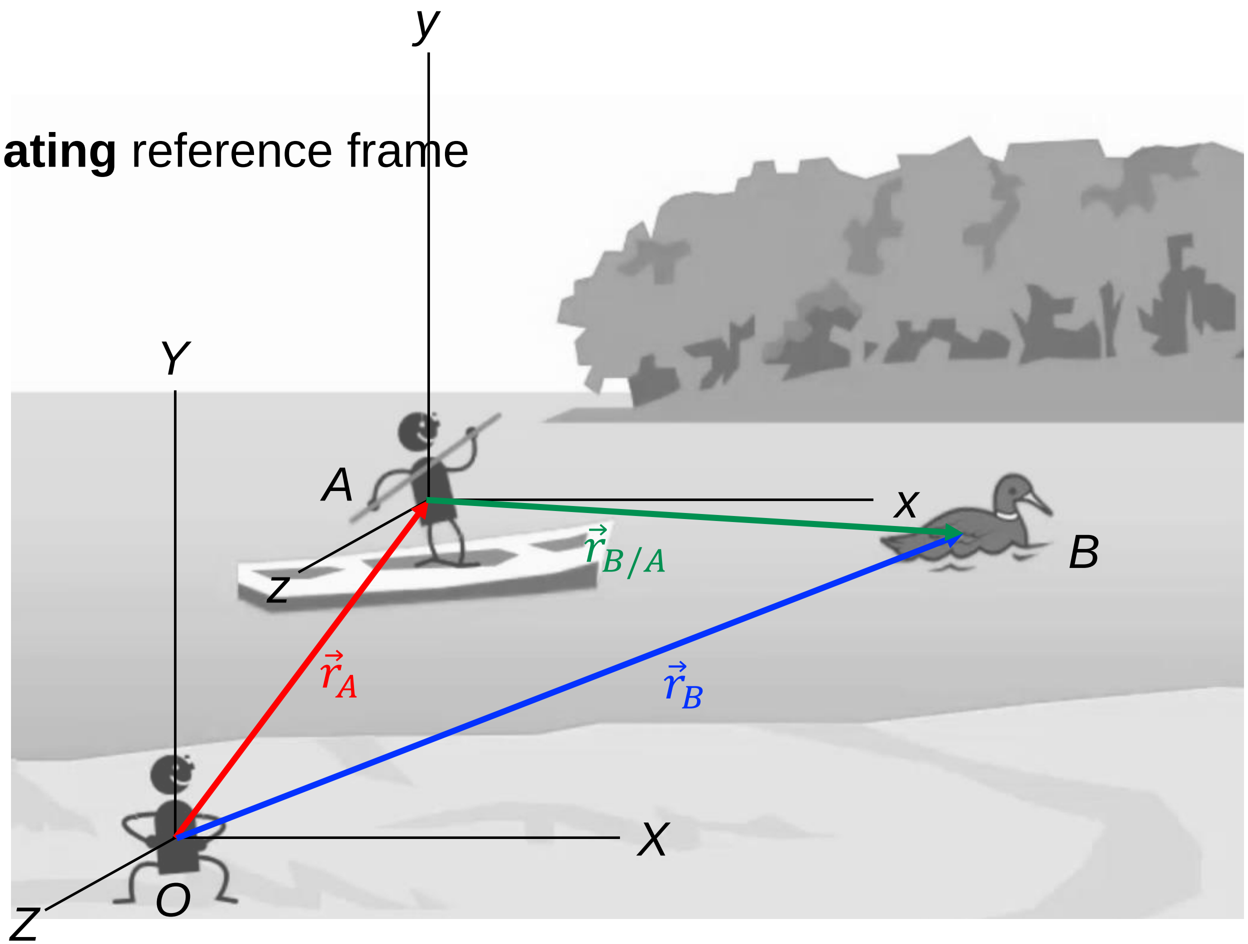
- References
  - Engineering Mechanics: Dynamics 12th Edition, RC Hibbeler, Prentice Hall

Contents

- Relative Motion with Translating Frame
- Relative Motion using Rotating Axes
- Example Calculation

Recall: Relative motion with translating frame

Translating reference frame

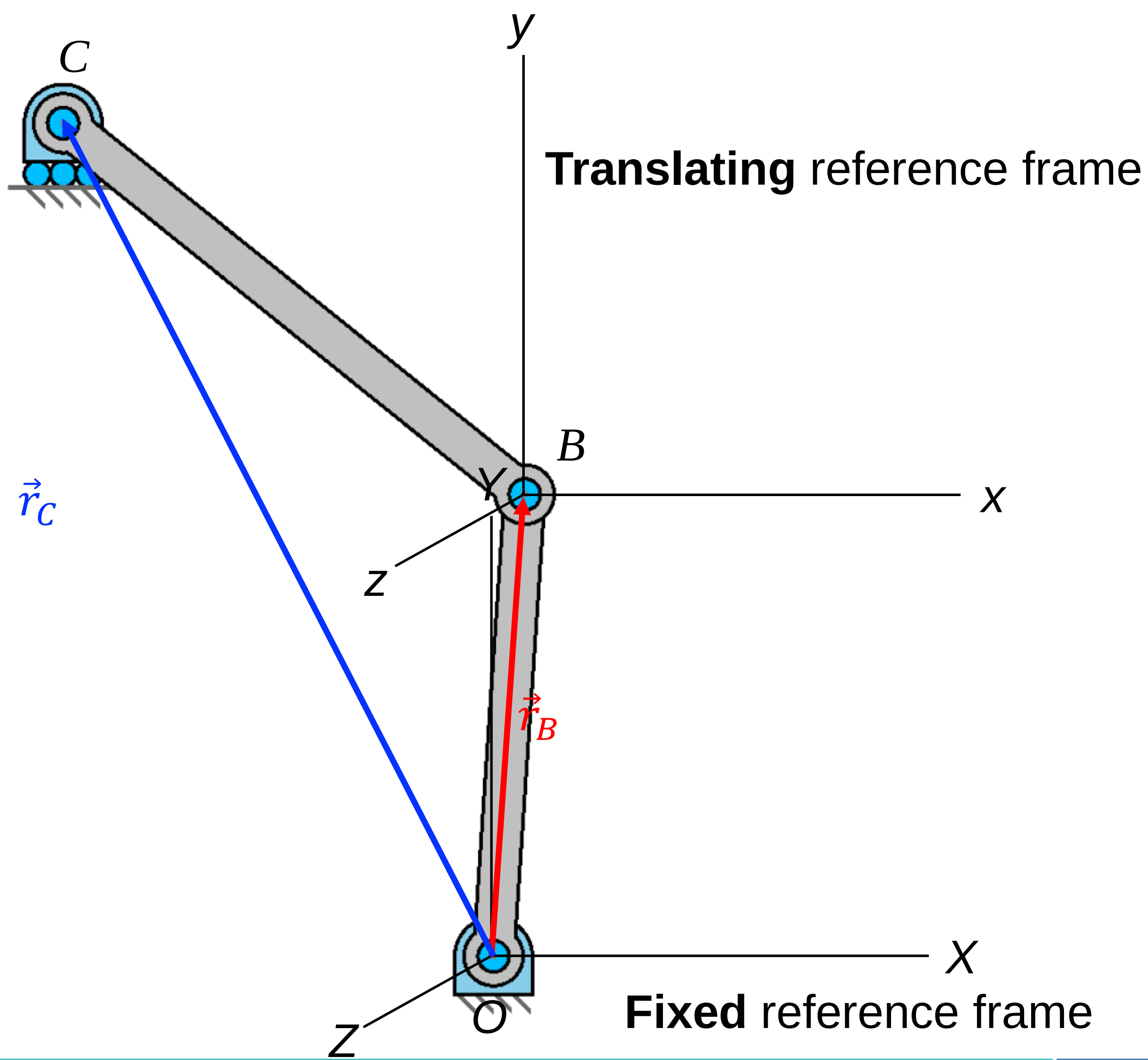


Fixed reference frame

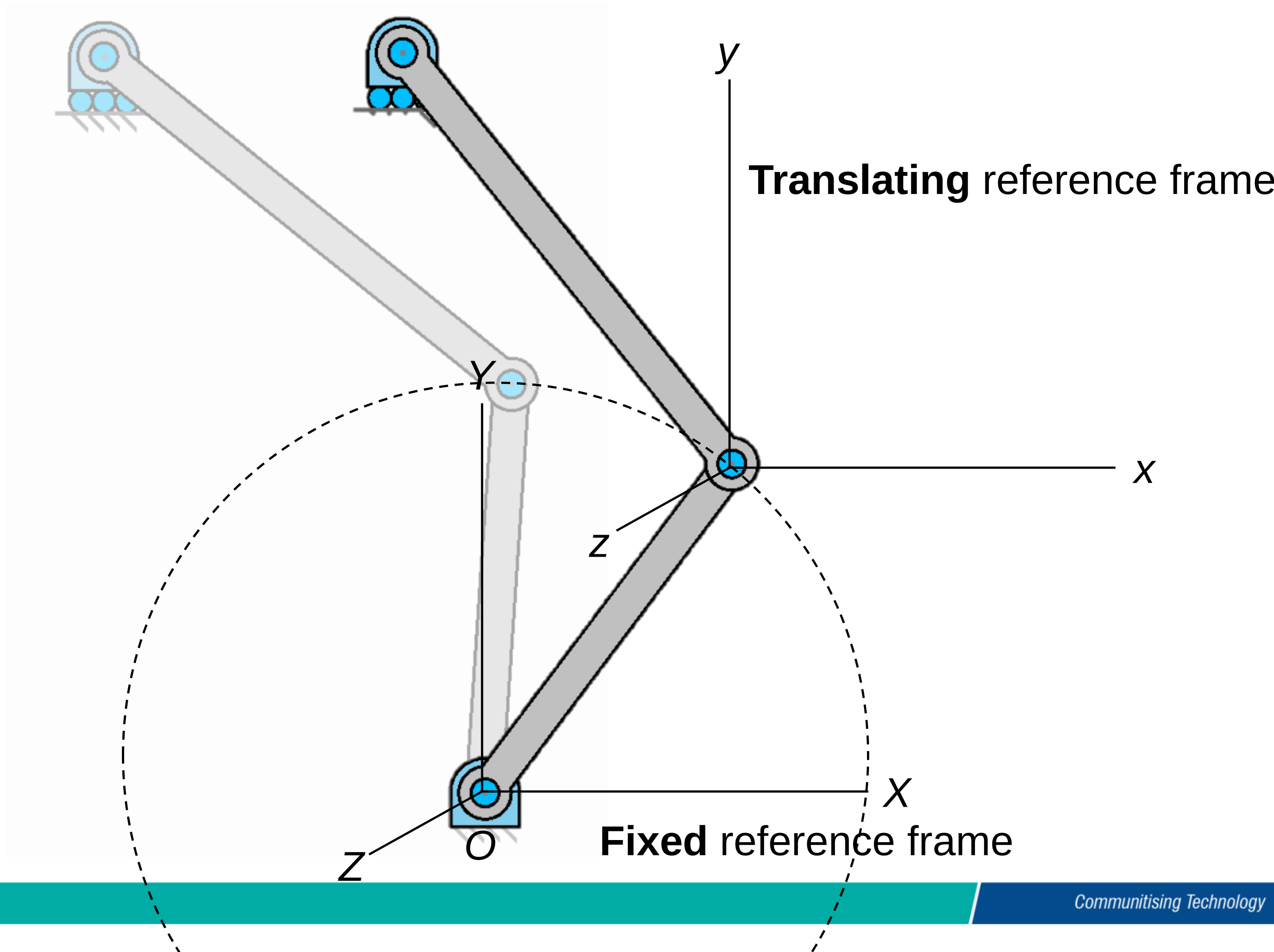
$\vec{r}_A$ and $\vec{r}_B$ are absolute positions. $\vec{r}_{B/A}$ are the relative position of *B* relative to *A*.

$$\vec{r}_{B/A} = \vec{r}_B - \vec{r}_A$$

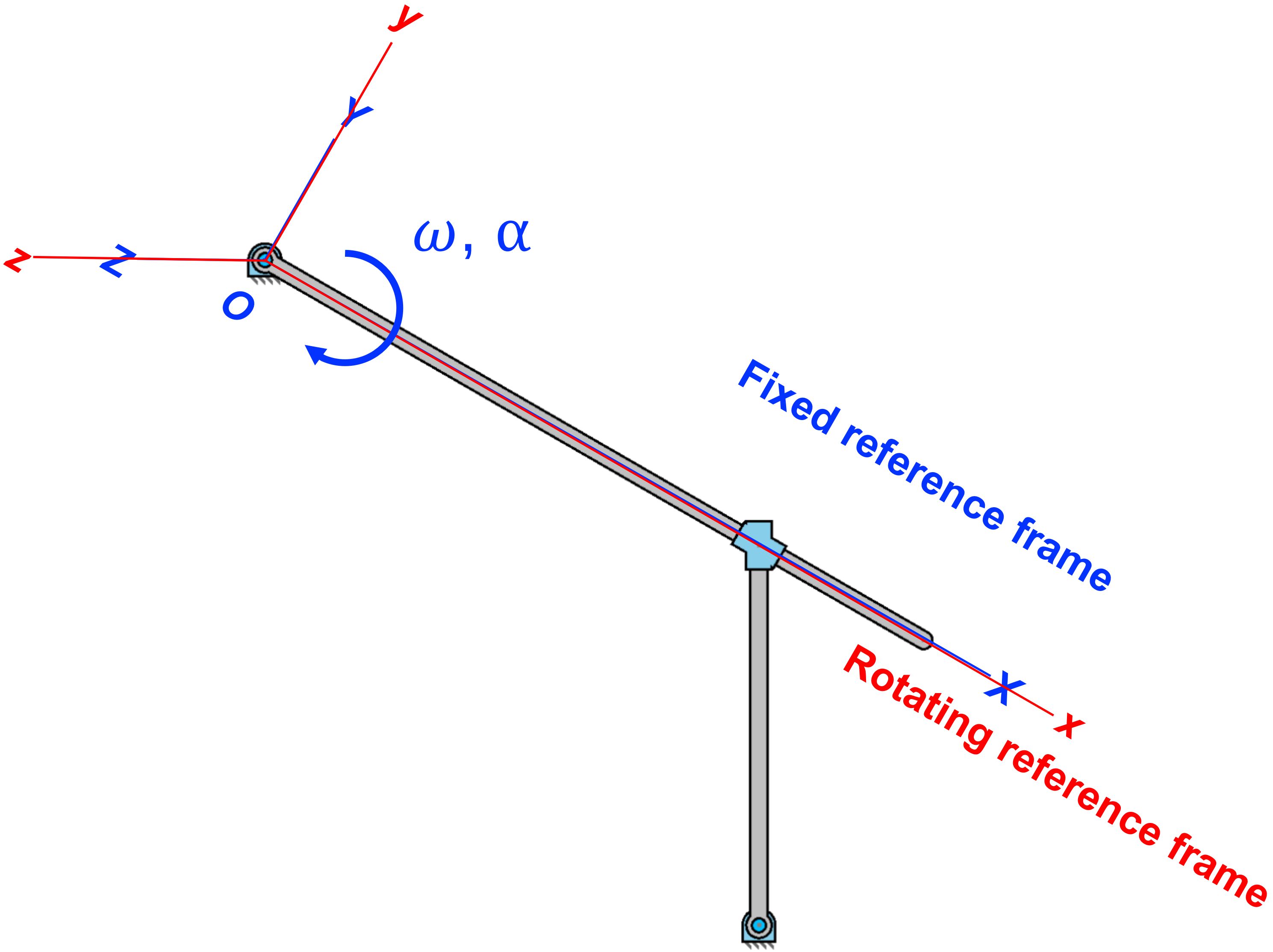
Recall: Relative motion with translating frame



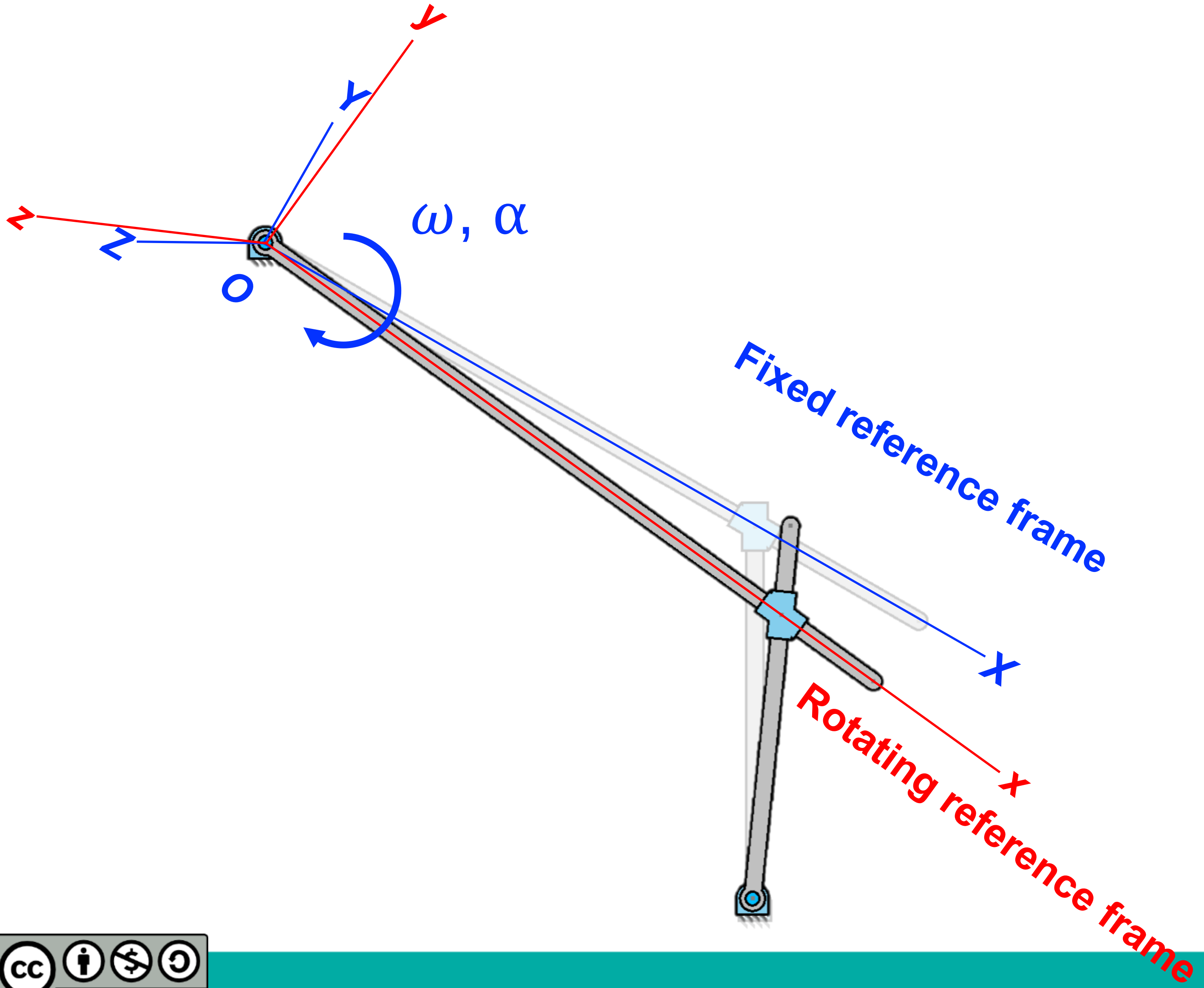
Recall: Relative motion with translating frame



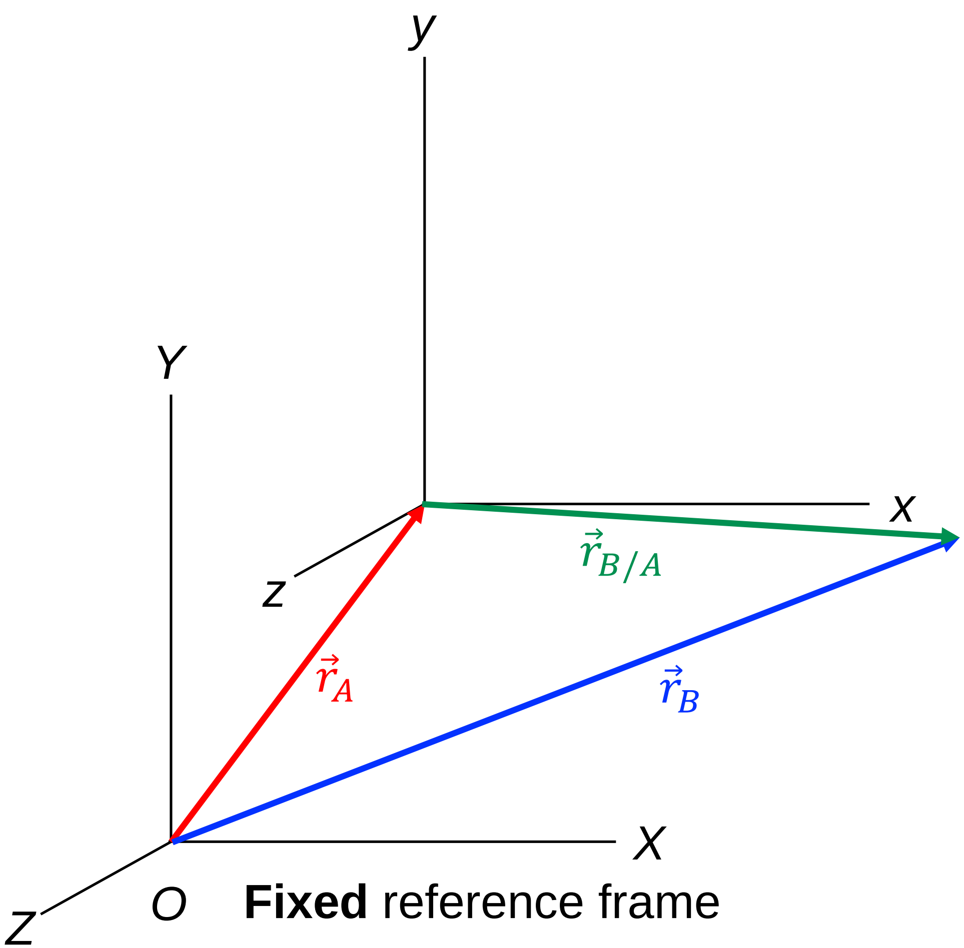
Relative Motion – Rotating Axes



Relative Motion – Rotating Axes



Relative Motion – Rotating Axes



Relative position:

$$\vec{r}_{B/A} = \vec{r}_B - \vec{r}_A$$

Relative velocity:

$$\vec{v}_{B/A} = \vec{v}_B - \vec{v}_A$$

Relative acceleration:

$$\vec{a}_{B/A} = \vec{a}_B - \vec{a}_A$$

Remember: For translating axes, the unit vector $\hat{\imath}$, $\hat{\jmath}$, $\hat{k}$ do not change with time.

$$\vec{r}_{B/A} = x\hat{i} + y\hat{j} + z\hat{k}$$

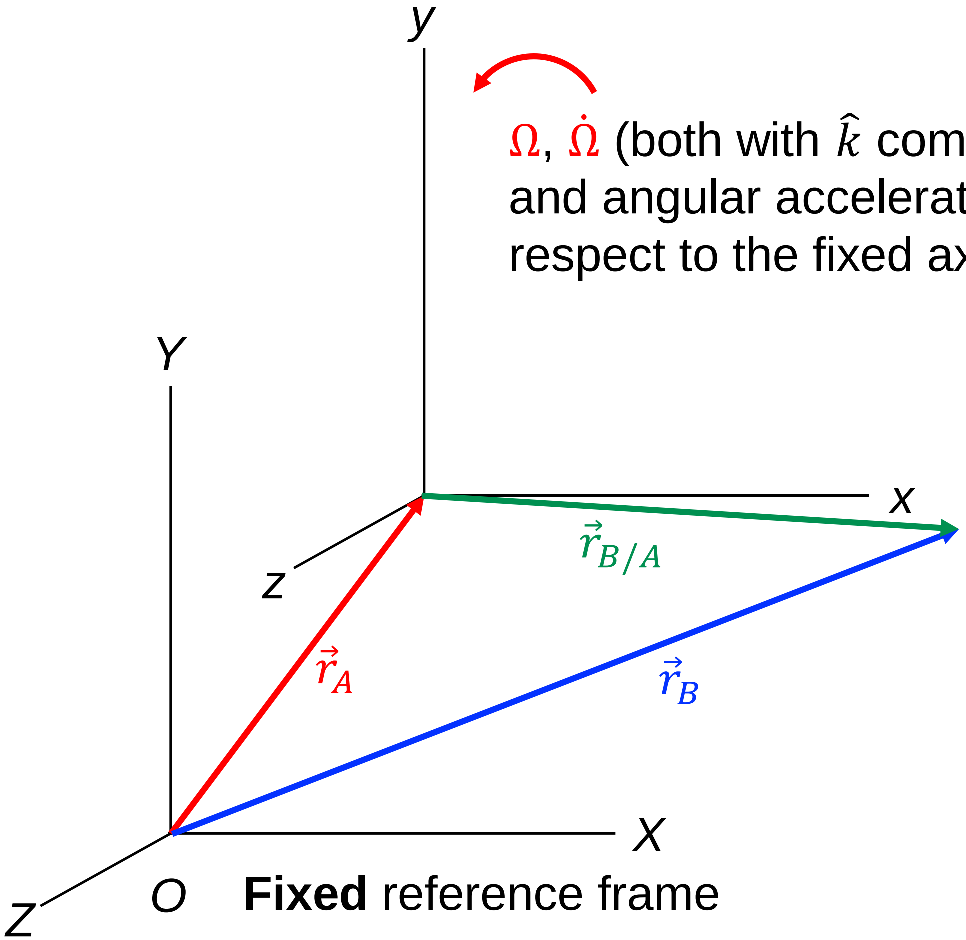
$$\vec{v}_{B/A} = \frac{d\vec{r}_{B/A}}{dt} = \frac{dx}{dt}\hat{i} + \frac{dy}{dt}\hat{j} + \frac{dz}{dt}\hat{k}$$

Only *x*, *y* and *z* are **functions of time**.
$\hat{\imath}$, $\hat{\jmath}$, $\hat{k}$ are **not** functions of time.

Relative Motion – Rotating Axes

Rotating reference frame (maybe **translating** too)

Ω, $\dot{\Omega}$ (both with $\hat{k}$ component) are the angular velocity and angular acceleration of the rotating axes with respect to the fixed axes



$$\vec{r}_{B/A} = \vec{r}_B - \vec{r}_A$$

$$\frac{d\vec{r}_{B/A}}{dt} = \vec{v}_B - \vec{v}_A$$

Relative Motion – Rotating Axes

$$\vec{r}_{B/A} = \vec{r}_B - \vec{r}_A$$

$$\frac{d\vec{r}_{B/A}}{dt} = \vec{v}_B - \vec{v}_A$$

$$\vec{r}_{B/A} = x\hat{i} + y\hat{j}$$

$\hat{\imath}$ and $\hat{\jmath}$ are now variables.
To integrate $\vec{r}_{B/A}$, **chain rule** need to be used.

$$d(uv) = du\ v + u\ dv$$

$$\frac{d\vec{r}_{B/A}}{dt} = \frac{dx}{dt}\hat{i} + x\frac{d\hat{i}}{dt} + \frac{dy}{dt}\hat{j} + y\frac{d\hat{j}}{dt}$$

Relative Motion – Rotating Axes

$$\vec{r}_{B/A} = \vec{r}_B - \vec{r}_A$$

$$\frac{d\vec{r}_{B/A}}{dt} = \vec{v}_B - \vec{v}_A$$

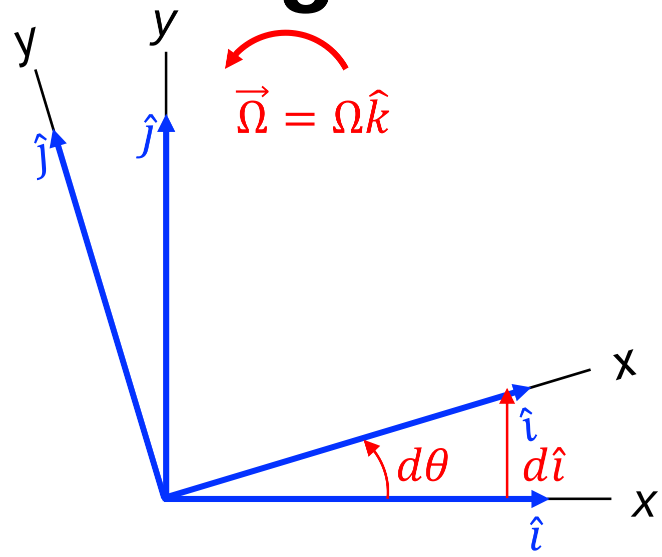
$$\vec{r}_{B/A} = x\hat{i} + y\hat{j}$$

$$\frac{d\vec{r}_{B/A}}{dt} = \frac{dx}{dt}\hat{i} + x\frac{d\hat{i}}{dt} + \frac{dy}{dt}\hat{j} + y\frac{d\hat{j}}{dt}$$

rearrange:

$$\frac{d\vec{r}_{B/A}}{dt} = \underbrace{\frac{dx}{dt}\hat{i} + \frac{dy}{dt}\hat{j}}_{} + \underbrace{x\frac{d\hat{i}}{dt} + y\frac{d\hat{j}}{dt}}_{}$$

Velocity of Point *B* as observed at Point *A* in *xy* frame $(\vec{v}_{B/A})_{xy}$



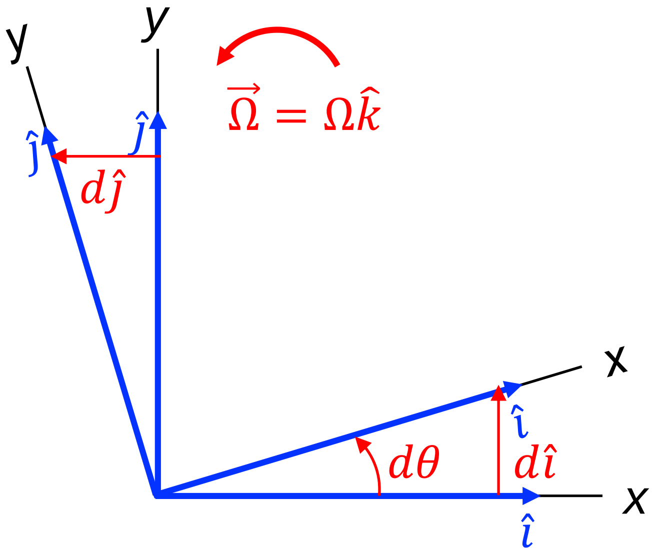
From $s = r\theta$, assuming $\hat{i}$ and $\hat{j} = 1$

$$d\hat{i} = (1 \cdot d\theta)\hat{j}$$

$$\frac{d\hat{i}}{dt} = \frac{d\theta}{dt}\hat{j} \qquad \text{where } \frac{d\theta}{dt} = \Omega$$

$$\therefore \frac{d\hat{i}}{dt} = \vec{\Omega} \times \hat{i}$$

Relative Motion – Rotating Axes



$$\frac{d\vec{r}_{B/A}}{dt} = \vec{v}_B - \vec{v}_A$$

$$\frac{d\vec{r}_{B/A}}{dt} = \frac{dx}{dt}\hat{i} + \frac{dy}{dt}\hat{j} + x\frac{d\hat{i}}{dt} + y\frac{d\hat{j}}{dt}$$

$$x(\vec{\Omega} \times \hat{i}) + y(\vec{\Omega} \times \hat{j})$$

$$\vec{\Omega} \times (x\hat{i} + y\hat{j}) = \vec{\Omega} \times \vec{r}_{B/A}$$

From $s = r\theta$, assuming $\hat{i}$ and $\hat{j} = 1$

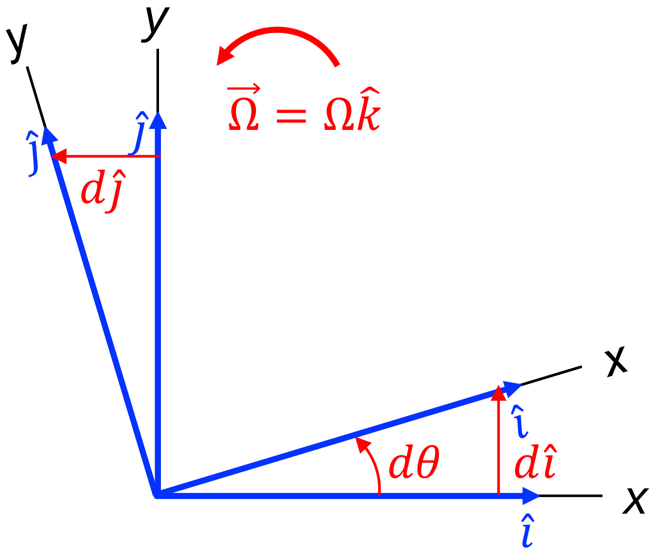
$$d\hat{i} = (1 \cdot d\theta)\hat{j}$$

$$\frac{d\hat{i}}{dt} = \frac{d\theta}{dt}\hat{j} \qquad \text{where } \frac{d\theta}{dt} = \Omega$$

$$\therefore \frac{d\hat{i}}{dt} = \vec{\Omega} \times \hat{i} \qquad \frac{d\hat{j}}{dt} = \frac{d\theta}{dt}(-\hat{i}) = \vec{\Omega} \times \hat{j}$$

$$\boxed{\vec{v}_B = \vec{v}_A + \vec{\Omega} \times \vec{r}_{B/A} + (\vec{v}_{B/A})_{xy}}$$

Relative Motion – Rotating Axes



$$\vec{v}_B = \vec{v}_A + \vec{\Omega} \times \vec{r}_{B/A} + (\vec{v}_{B/A})_{xy}$$

Differentiate to get the equation for acceleration

$$\vec{a}_B = \vec{a}_A + \frac{d\vec{\Omega}}{dt} \times \vec{r}_{B/A} + \vec{\Omega} \times \frac{d\vec{r}_{B/A}}{dt} + \frac{d\vec{v}_{B/A})_{xy}}{dt}$$

$$\frac{d\vec{r}_{B/A}}{dt} = \vec{\Omega} \times \vec{r}_{B/A} + (\vec{v}_{B/A})_{xy}$$

$$\frac{d\vec{v}_{B/A}}{dt} = \vec{\Omega} \times (\vec{v}_{B/A})_{xy} + (\vec{a}_{B/A})_{xy}$$

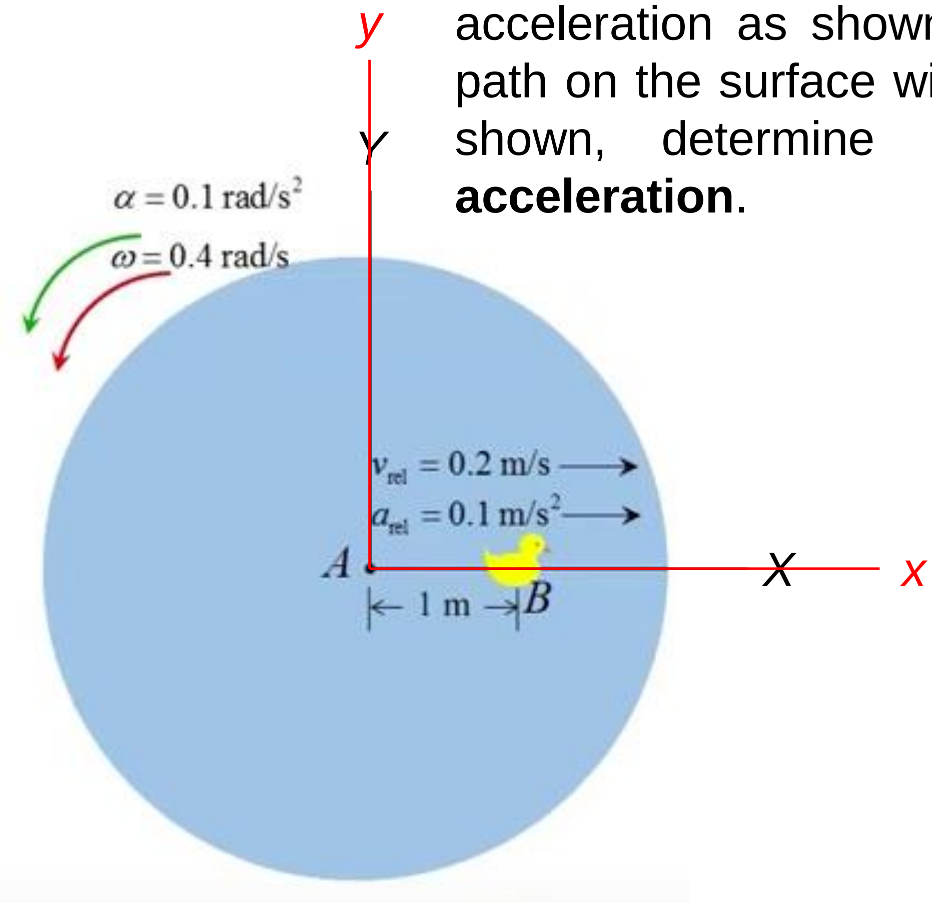
$$\vec{a}_B = \vec{a}_A + \dot{\Omega} \times \vec{r}_{B/A} + \vec{\Omega} \times (\vec{\Omega} \times \vec{r}_{B/A} + (\vec{v}_{B/A})_{xy}) + \vec{\Omega} \times (\vec{v}_{B/A})_{xy} + (\vec{a}_{B/A})_{xy}$$

$$\vec{a}_B = \vec{a}_A + \dot{\Omega} \times \vec{r}_{B/A} + \vec{\Omega} \times (\vec{\Omega} \times \vec{r}_{B/A}) + 2\vec{\Omega} \times (\vec{v}_{B/A})_{xy} + (\vec{a}_{B/A})_{xy}$$

$$\vec{a}_B = \vec{a}_A + \dot{\vec{\Omega}} \times \vec{r}_{B/A} - \Omega^2 \vec{r}_{B/A} + 2\vec{\Omega} \times (\vec{v}_{B/A})_{xy} + (\vec{a}_{B/A})_{xy}$$

Example Calculation

The horizontal surface rotates with the angular velocity and acceleration as shown. If a toy duck moves along the straight path on the surface with the relative velocity and acceleration as shown, determine its **absolute velocity** and **absolute acceleration**.



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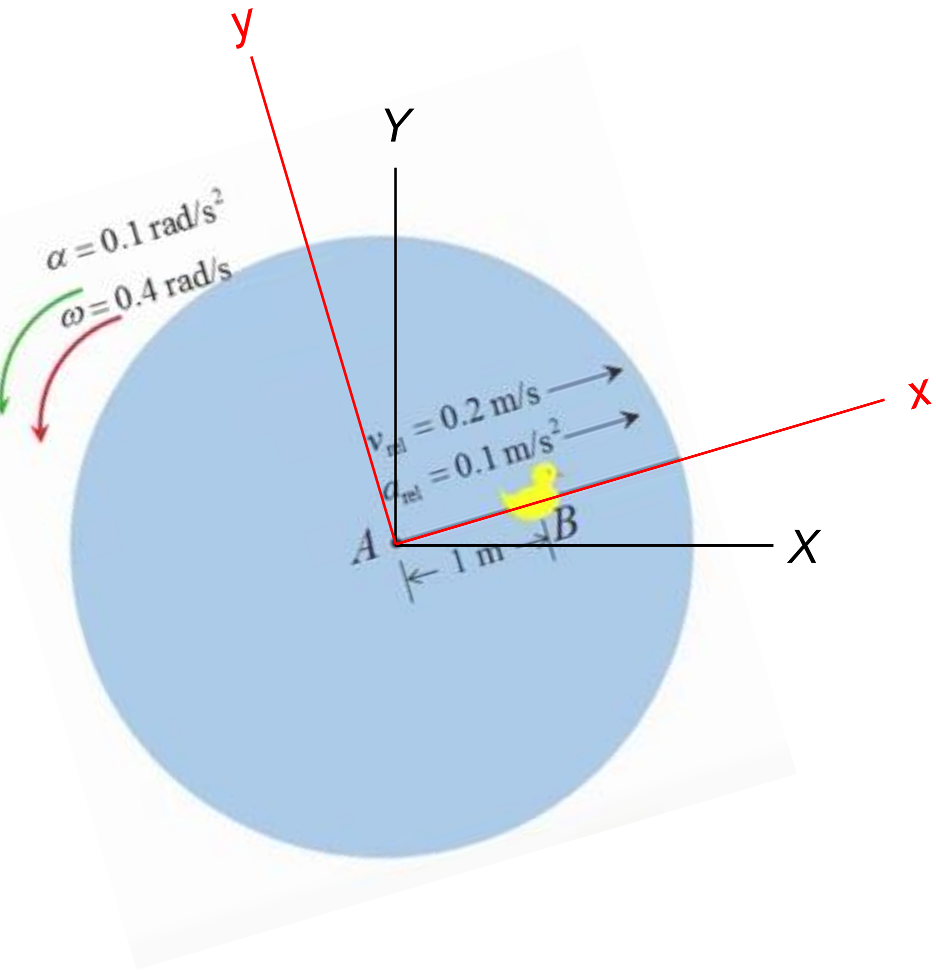
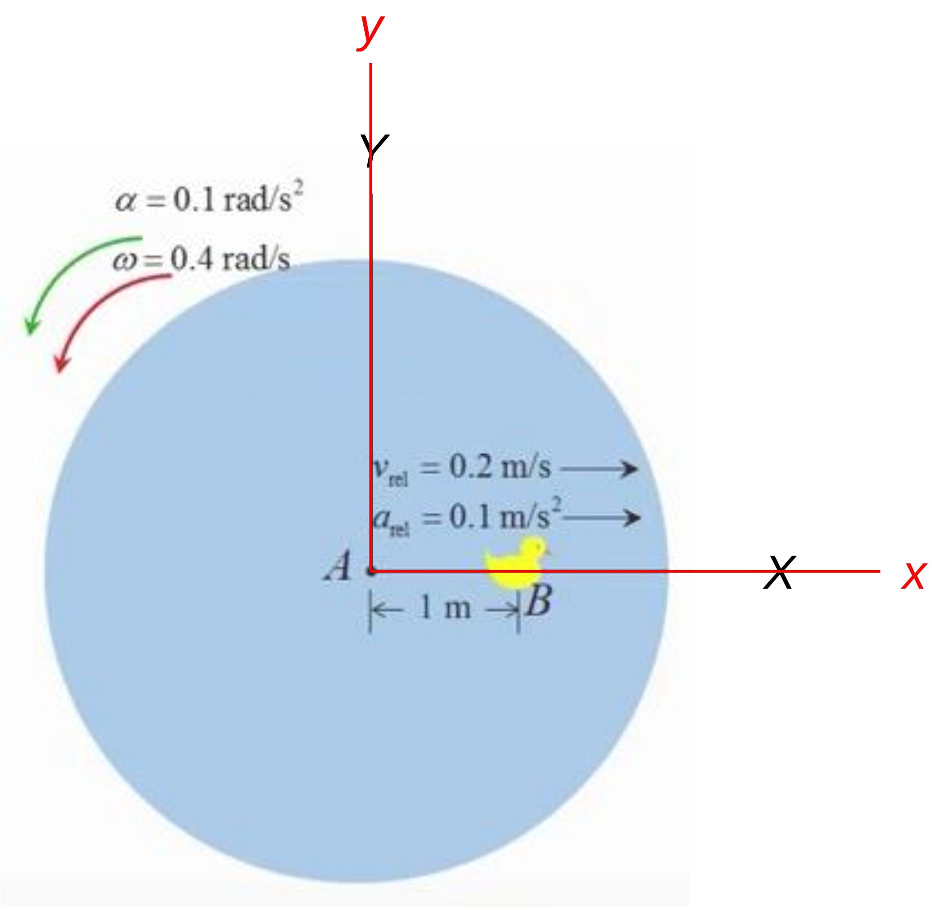
Example Calculation



Example Calculation



Absolute velocity of the toy duck = $\vec{v}_B$

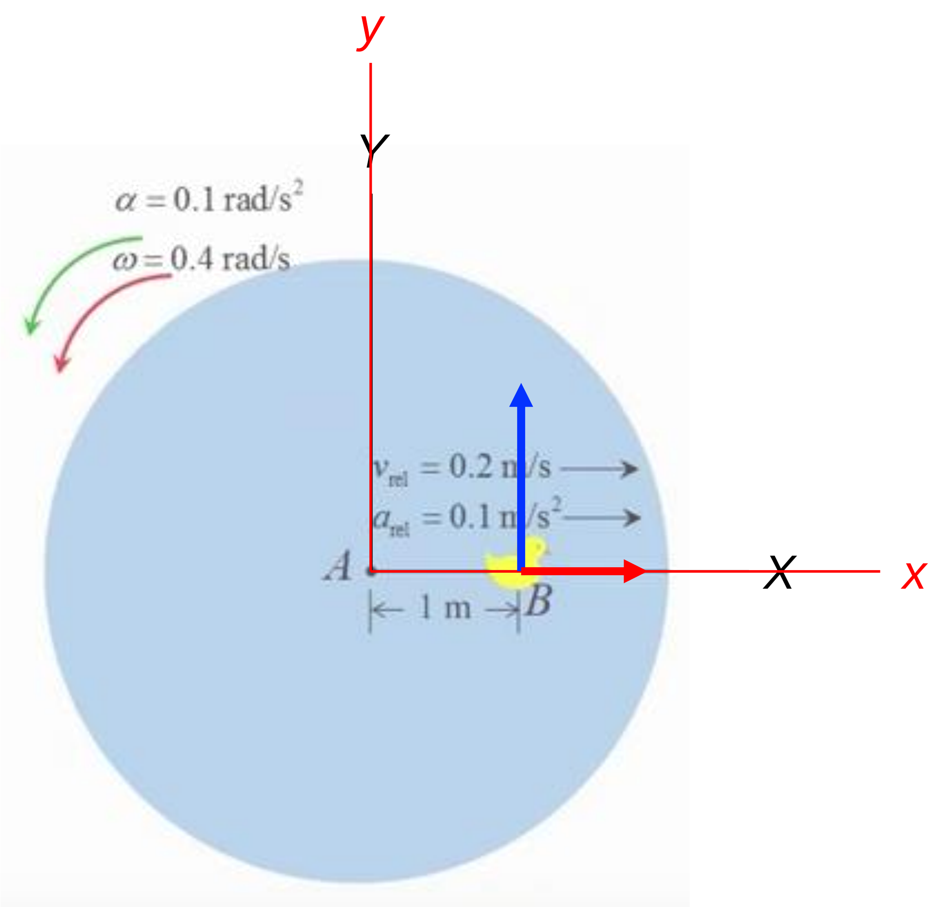
$$\vec{v}_B = \vec{v}_A + \vec{\Omega} \times \vec{r}_{B/A} + (\vec{v}_{B/A})_{xy}$$

0 $\quad 0.4\hat{k}$ $\quad 1\hat{\imath}$ $\quad 0.2\hat{\imath}$

$$\vec{v}_B = 0.4\hat{k} \times 1\hat{i} + 0.2\hat{i}$$
$$= 0.4\hat{j} + 0.2\hat{i}$$

$$\vec{v}_B = 0.2\hat{i} + 0.4\hat{j} \text{ m/s}$$

Example Calculation



$$\vec{v}_B = \vec{v}_A + \vec{\Omega} \times \vec{r}_{B/A} + (\vec{v}_{B/A})_{xy}$$

Absolute velocity of Point *B* on the *xy* frame.

Motion of *xy* frame observed from the *XY* frame

Relative velocity of Point *B* wrt Point *A*

Motion of Point *B* observed from the *xy* frame

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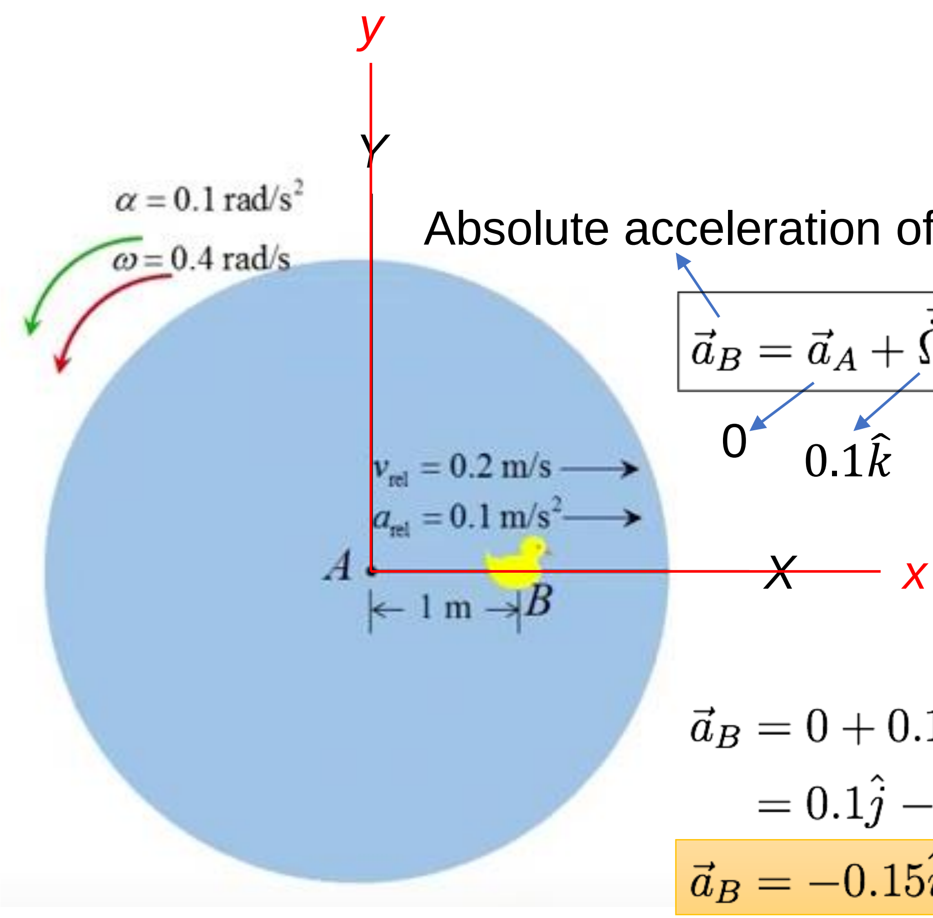
Example Calculation



Absolute acceleration of the toy duck

$$\vec{a}_B = \vec{a}_A + \dot{\vec{\Omega}} \times \vec{r}_{B/A} - \Omega^2 \vec{r}_{B/A} + 2\vec{\Omega} \times (\vec{v}_{B/A})_{xy} + (\vec{a}_{B/A})_{xy}$$

0 | $0.1\hat{k}$ | $1\hat{\imath}$ | 0.4^2 | $1\hat{\imath}$ | $2(0.4\hat{k})$ | $0.2\hat{\imath}$ | $0.1\hat{\imath}$

$$\vec{a}_B = 0 + 0.1\hat{k} \times 1\hat{\imath} - (0.4^2)(1\hat{\imath}) + 2(0.4\hat{k}) \times 0.2\hat{\imath} + 0.1\hat{\imath}$$

$$= 0.1\hat{\jmath} - 0.16\hat{\imath} + 0.16\hat{\jmath} + 0.1\hat{\imath}$$

$$\vec{a}_B = -0.15\hat{\imath} + 0.26\hat{\jmath}\ \text{m/s}^2$$

$$\vec{a}_B = \vec{a}_A + \dot{\vec{\Omega}} \times \vec{r}_{B/A} - \Omega^2 \vec{r}_{B/A} + 2\vec{\Omega} \times (\vec{v}_{B/A})_{xy} + (\vec{a}_{B/A})_{xy}$$

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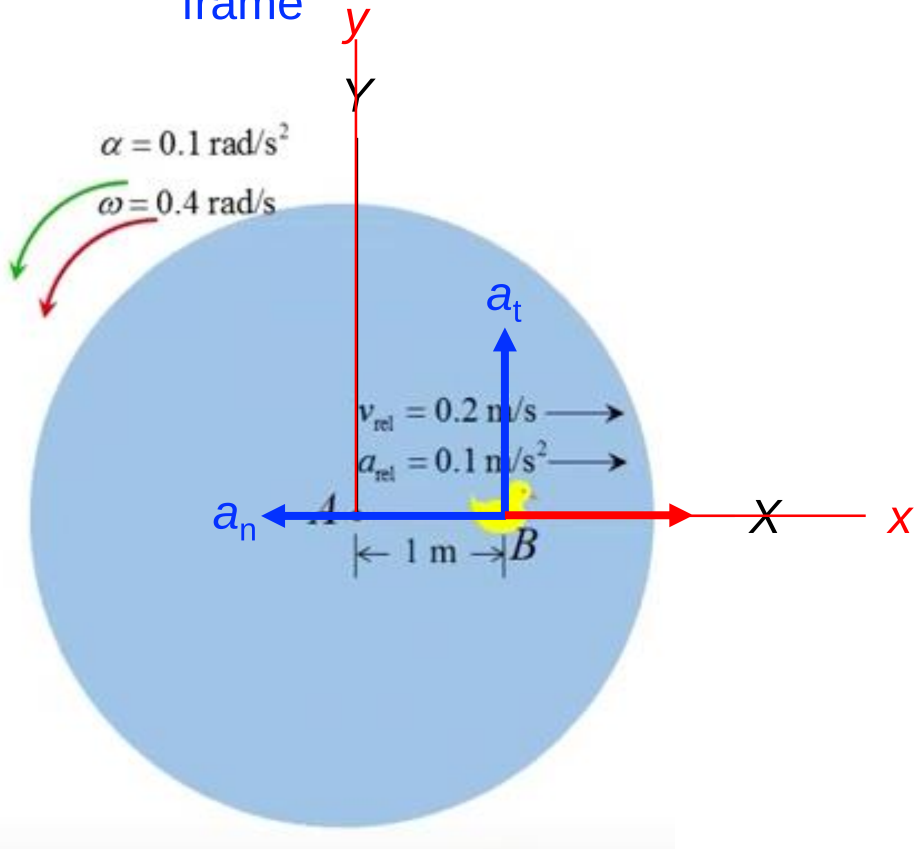
Absolute acceleration of Point *B* on the *xy* frame.

Motion of *xy* frame observed from the *XY* frame

Coriolis acceleration

Relative acceleration of Point *B* wrt Point *A*

Motion of Point *B* observed from the *xy* frame



An important mathematical correction factor that must be included when using rotating reference frame.

Named after a French engineer, **G.C. Coriolis** who was the first to determine it.

Conclusions

- Relative motion analysis using rotating axes is used to solve problems involving:
  - connected members that slide relative to one another, or
  - points that are not located on the same body.
- In this case, there is a new term called the Coriolis acceleration, which is given by $2\vec{\Omega} \times (\vec{v}_{B/A})_{xy}$.

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Planar Kinematics of a Rigid Body (Relative Motion using Rotating Axes)

"A man may imagine things that are false, but he can only understand things that are true."

– *Sir Isaac Newton*

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