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DYNAMICS

Planar Kinematics of a Rigid Body (Translation and Rotation)

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Translation and Rotation

- Aims
  - To revise on the position vector.
  - To introduce about the types of planar motion of rigid body.
  - To discuss about translation and rotation about a fixed axis.
  - To analyse the motion of a particular point on a rigid body.
- Expected Outcomes
  - Students are able to determine the position vector of a point on a rigid body.
  - Students are able to distinguish different types of planar motion of a rigid body.
  - Students are able to calculate the velocity and acceleration of a particular point on a rigid body.
- References
  - Engineering Mechanics: Dynamics 12th Edition, RC Hibbeler, Prentice Hall

Content

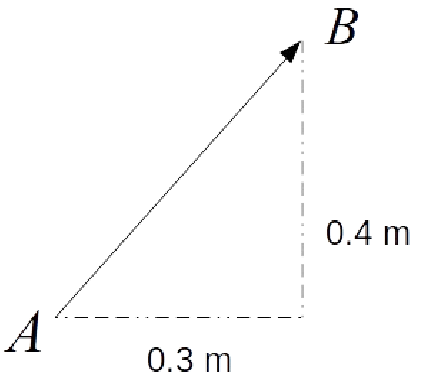
- Introduction
- Position Vector
- Planar Motion of a Rigid Body
- Translation
- Rotation about a Fixed Axis
- Conclusions

Introduction

- **Engineering Mechanics**
  - **Rigid Body**
    - **Statics**
    - **Dynamics**
      - Kinematics
        - Planar Kinematics - Translation
        - Planar Kinematics – Rotation about a fixed axis
        - Planar Kinematics – General plane motion
      - Kinetics
        - Planar Kinetics – Force and Acceleration
        - Planar Kinetics – Work and Energy
        - Planar Kinetics – Impulse and Momentum
  - **Deformable Body**
  - **Fluid Mechanics**

Position Vector

Position vector is a vector that represents a directed line between two points. Graphically, it is an arrow with head and tail. For example, referring to the mechanism below we represent the position vector $\vec{r}_{B/A}$ as:



$$\vec{r}_{B/A} = 0.3\,\hat{\imath} + 0.4\hat{\jmath}\text{ m}$$

The $\hat{\imath}$ and $\hat{\jmath}$ are the unit vectors along the x and y axis respectively. We read the above equation like the following:

The position vector of point B relative to point A is 0.3 m to the positive x-axis plus 0.4 meter to the positive y-axis.

Position Vector

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Notice the difference between $\vec{r}_{A/B}$ with $\vec{r}_{B/A}$. It is opposite to each other. Therefore $\vec{r}_{A/B}$ is:

$$\vec{r}_{A/B} = -0.3\hat{\imath} - 0.4\hat{\jmath}$$

Always remember which one is the head and which one is the tail.

Head

$$\vec{r}_{B/A}$$

Tail

Sometimes, we use the angle of the vector and the distance between the points in order to determine the x and y components of the vector.

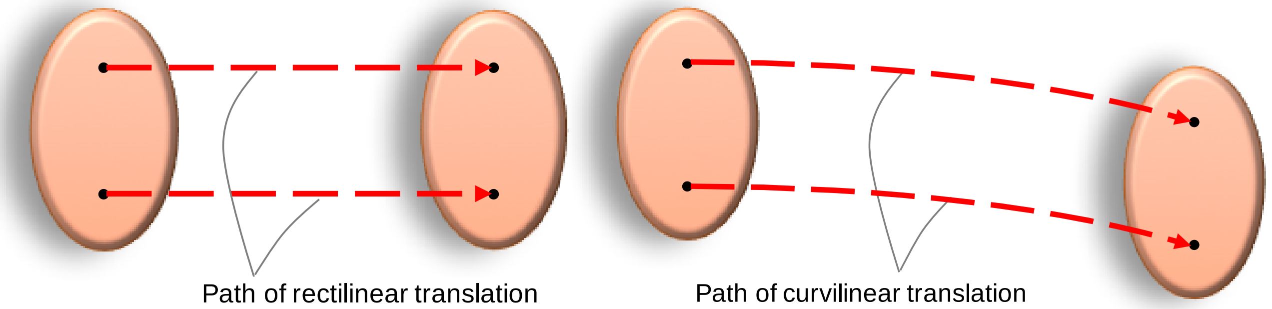
$$\vec{r}_{B/AB} = l_{AB}\cos\theta_1\hat{\imath} + l_{AB}\sin\theta_1\,\hat{\jmath}$$

In this practice, you must identify the position vectors on a mechanism. We must know the position vector in order to determine the velocity and acceleration of moving rigid bodies.

Planar Motion of a Rigid Body

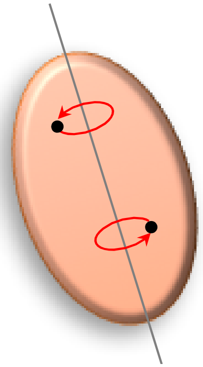
Translation

The path of 2 points remains parallel



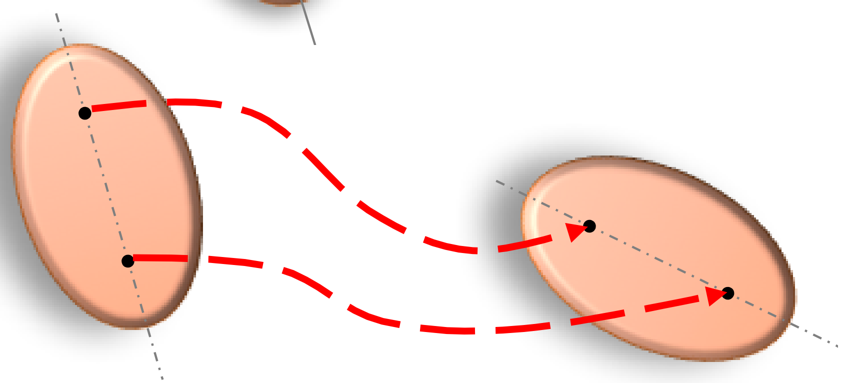
Rotation about a fixed axis

The body rotates about a fixed axis, all points on the body except for those on the line of rotation moves along a circular path.

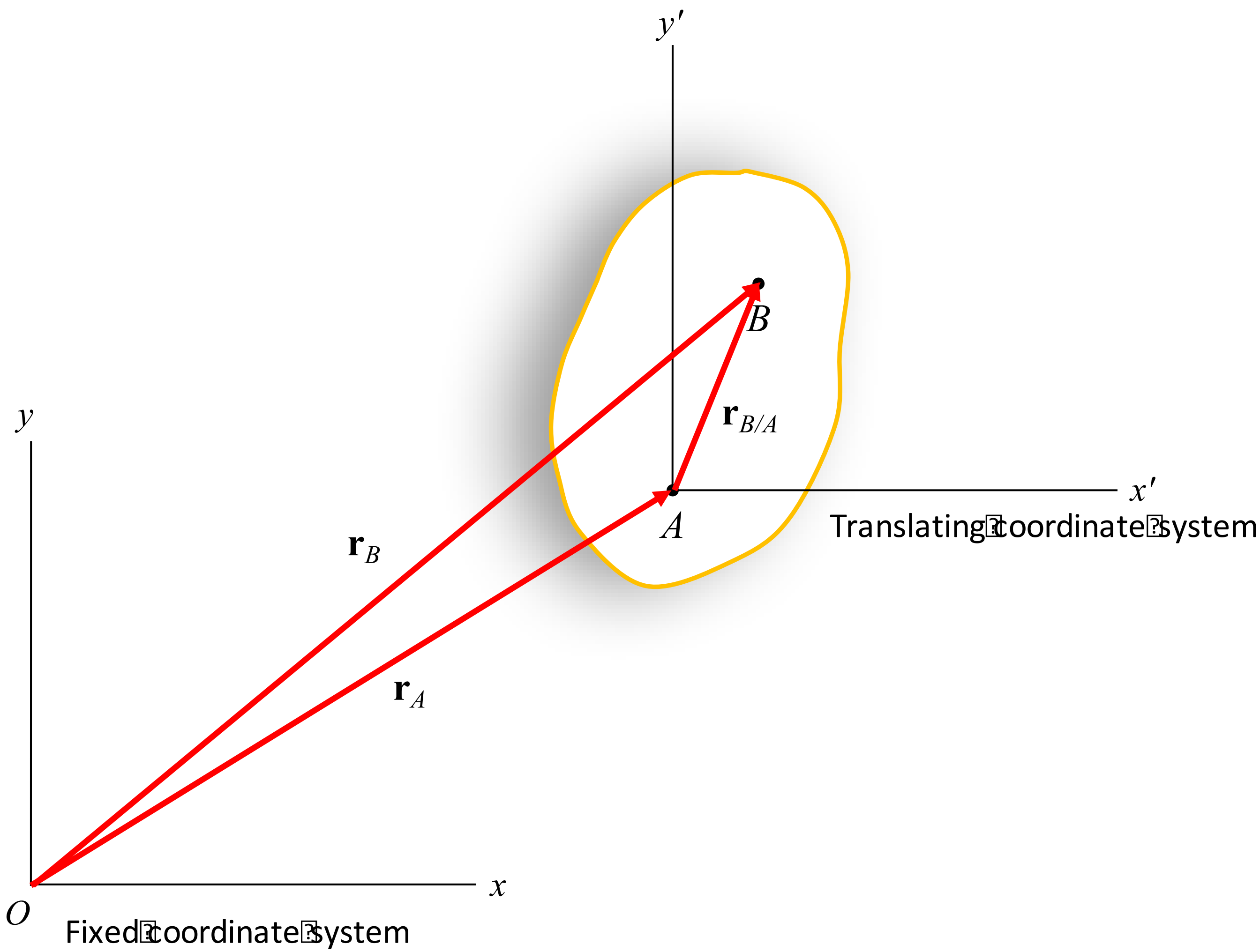


General plane motion

Combination of translation and rotation



Translation

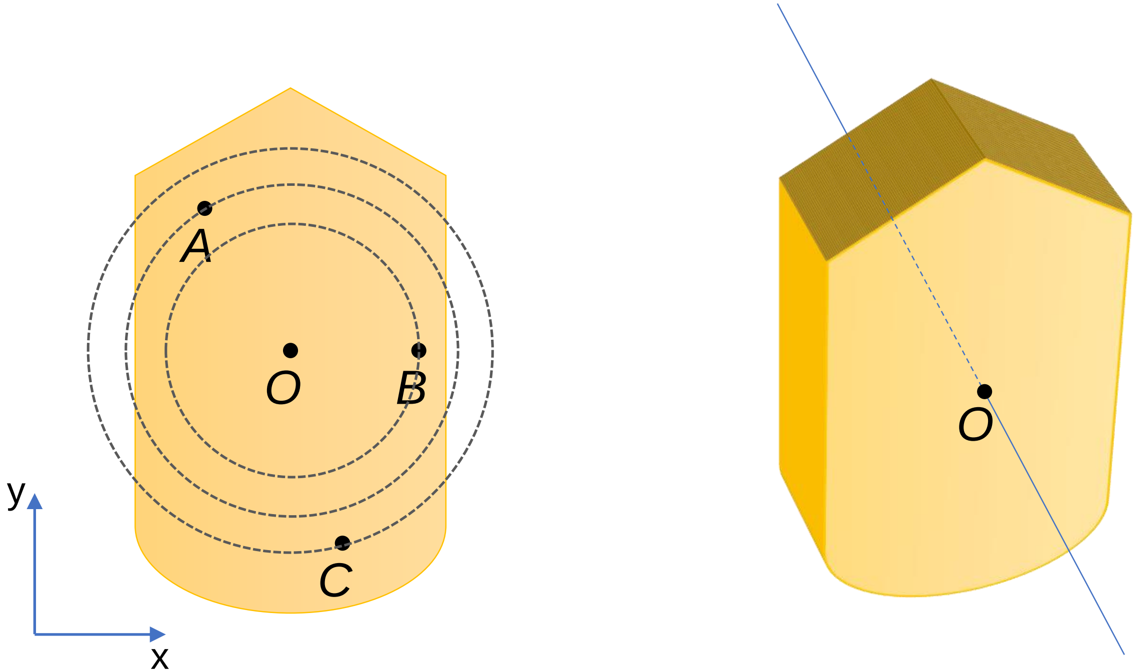


$$\vec{r}_B = \vec{r}_A + \vec{r}_{B/A}$$

$$\vec{v}_B = \vec{v}_A$$

$$\vec{a}_B = \vec{a}_A$$

Rotation about a fixed axis



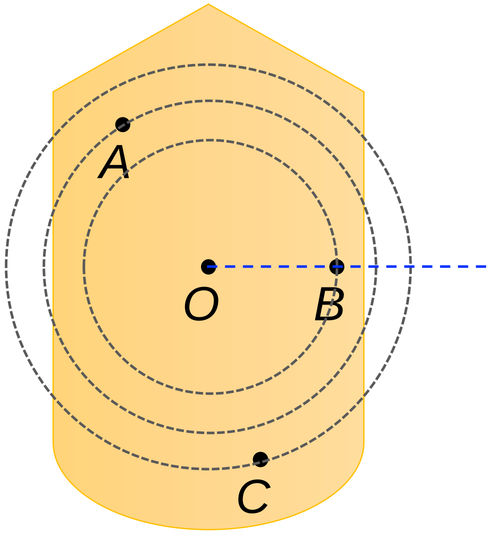
For rigid body planar motion, rotation is limited to the *x*-*y* plane.

The rigid body is actually rotating about *z*-axis that passes through Point *O*.

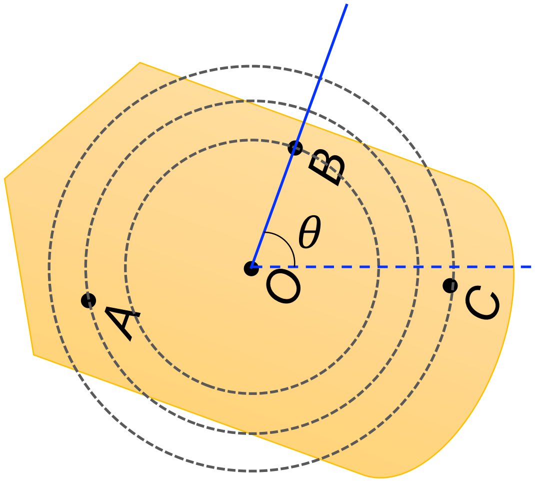
That is why this motion is called **rotation about a fixed axis**.

The path of motion of any point (particle) on the rigid body is **ALWAYS CIRCULAR**

Rotation about a fixed axis



Rotation about a fixed axis



Linear motion	Angular motion
$v = \frac{ds}{dt}$	$\omega = \frac{d\theta}{dt}$
$a = \frac{dv}{dt}$	$\alpha = \frac{d\omega}{dt}$
$a\,ds = v\,dv$	$\alpha\,d\theta = \omega\,d\omega$

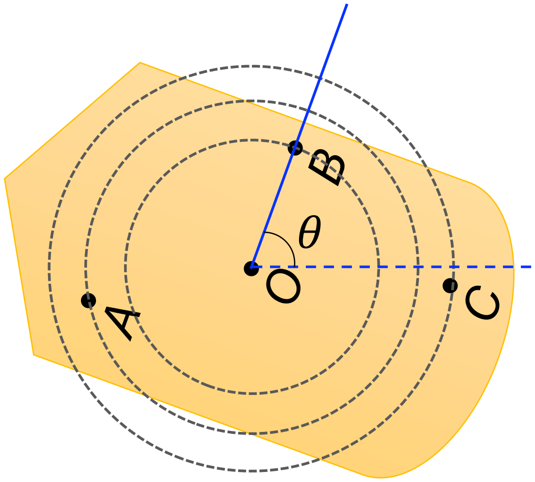
θ → angular position [rad]

$\frac{d\theta}{dt} = \omega$ → angular velocity [rad/s]

$\frac{d\omega}{dt} = \frac{d^2\theta}{dt^2} = \alpha$ → angular acceleration [rad/s^2]

$$\alpha\, d\theta = \omega\, d\omega$$

Rotation about a fixed axis

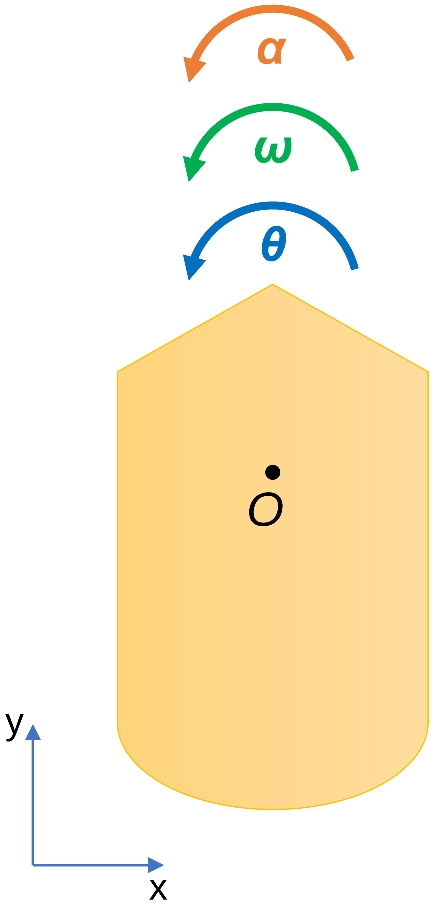


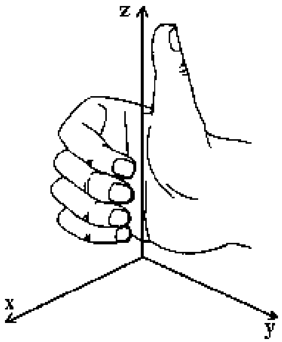
Linear motion	Angular motion
$v = \frac{ds}{dt}$	$\omega = \frac{d\theta}{dt}$
$a = \frac{dv}{dt}$	$\alpha = \frac{d\omega}{dt}$
$a ds = v dv$	$\alpha d\theta = \omega d\omega$

Motion with constant acceleration

Linear motion	Angular motion
$s = s_0 + v_0 t + \frac{1}{2} a_c t^2$	$\theta = \theta_0 + \omega_0 t + \frac{1}{2} \alpha_c t^2$
$v = v_0 + a_c t$	$\omega = \omega_0 + \alpha_c t$
$v^2 = v_0^2 + 2a_c(s - s_0)$	$\omega^2 = \omega_0^2 + 2\alpha_c(\theta - \theta_0)$

Rotation about a fixed axis





Counter-clockwise rotation = +ve

$$\vec{\theta} = \theta \hat{k}$$

$$\vec{\omega} = \omega \hat{k}$$

$$\vec{\alpha} = \alpha \hat{k}$$

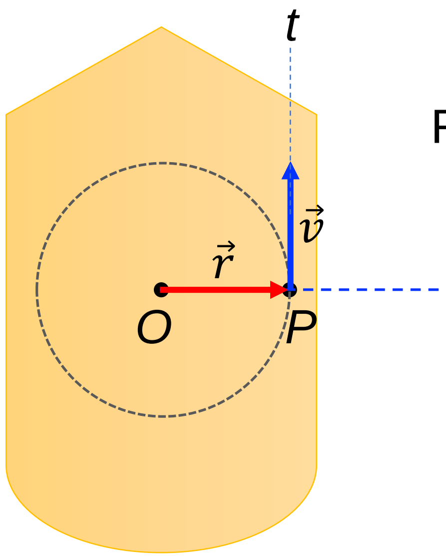
A **particle** can **NEVER ROTATE**.

In **rigid body rotation**, all **particles** undergo **curvilinear motion**.

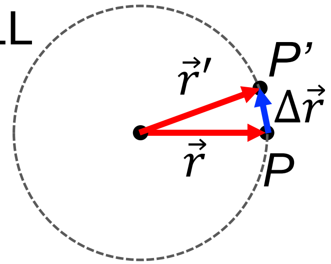
The **rigid body** has **angular velocity** and **angular acceleration**,
but the **particles (points)** have **linear velocity** and **linear acceleration**.

Curvilinear motion of Point *P*

The direction of $\vec{v}$ is ALWAYS TANGENT to the path



RECALL



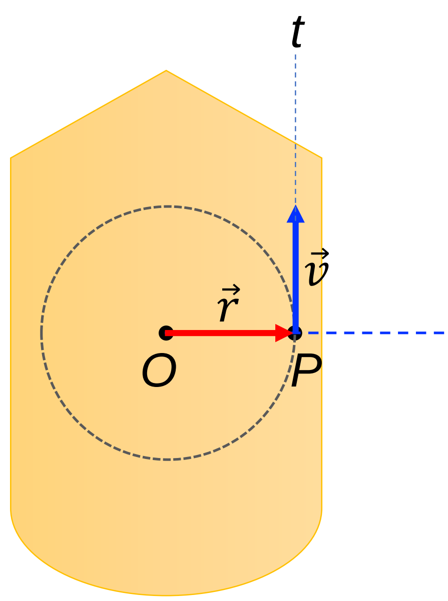
$$\vec{v} = \frac{\Delta\vec{r}}{\Delta t}$$

If $\Delta t \to 0$,
$\vec{r}$ and $\vec{r}'$ almost fall on the same line.

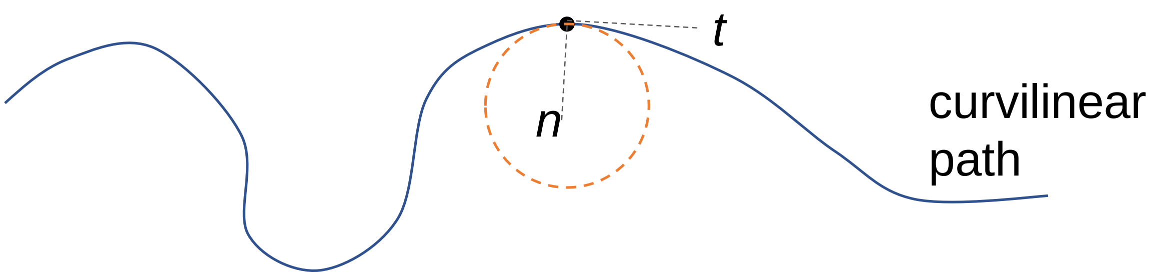
$d\vec{r}$ is perpendicular to $\vec{r}$,
→ $\mathrm{d}\vec{r}$ is tangent to the path.

∴ the direction of instantaneous velocity $\vec{v}$ is **always tangent** to the curved path.

Curvilinear motion of Point *P*



RECALL: tangential and normal component

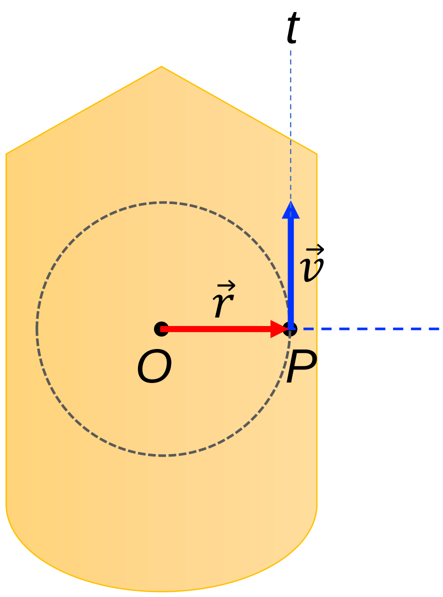


t : tangential axis – tangent to the path
n : normal axis – point towards the centre of the arc

Unit vector:

t-axis : $\hat{u}_t$
n-axis : $\hat{u}_n$

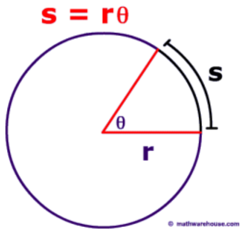
Curvilinear motion of Point *P*



Linear velocity of Point *P*

$$\vec{v} = v\hat{u}_t$$

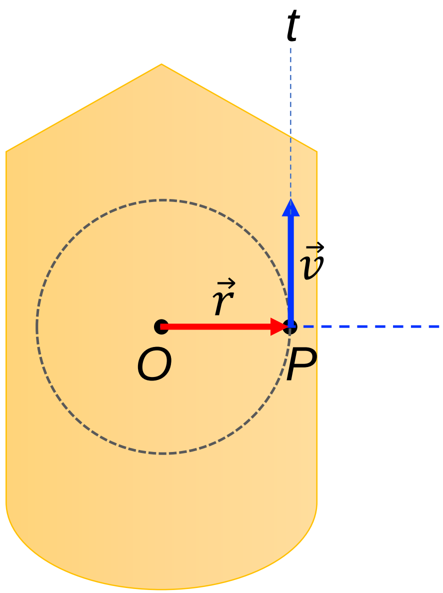
$$v = \frac{ds}{dt} \quad \text{where } s = r\theta$$



$$v = r\frac{d\theta}{dt} \quad \text{where } \frac{d\theta}{dt} = \omega$$

$$\therefore \vec{v} = \vec{\omega} \times \vec{r}$$

Curvilinear motion of Point *P*



Linear acceleration of Point *P*

$$\vec{a} = \frac{d\vec{v}}{dt} = \frac{d(\vec{\omega} \times \vec{r})}{dt}$$

Using product rule:

$$\frac{d}{dx}(uv) = u\frac{dv}{dx} + v\frac{du}{dx}$$

$$\vec{a} = \frac{d\vec{\omega}}{dt} \times \vec{r} + \vec{\omega} \times \frac{d\vec{r}}{dt}$$

$$= \vec{\alpha} \times \vec{r} + \vec{\omega} \times (\vec{\omega} \times \vec{r})$$

Use vector triple product:

$$\vec{a} \times (\vec{b} \times \vec{c}) = \vec{b}(\vec{a} \cdot \vec{c}) - \vec{c}(\vec{a} \cdot \vec{b})$$

$$\vec{\omega} \times (\vec{\omega} \times \vec{r}) = \vec{\omega}(\vec{\omega} \cdot \vec{r}) - \vec{r}(\vec{\omega} \cdot \vec{\omega})$$

with the angle between $\vec{r}$ and $\vec{\omega}$ → 90°

$$\vec{\omega} \times (\vec{\omega} \times \vec{r}) = \vec{\omega}(\omega r \cos 90°) - \vec{r}(\omega\omega \cos 0°)$$

$$\therefore \quad \vec{a} = \underbrace{\vec{\alpha} \times \vec{r}}_{a_t} - \underbrace{\omega^2 \vec{r}}_{a_n}$$

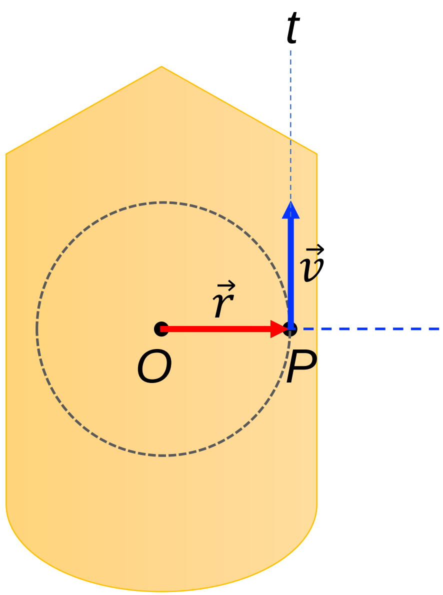
a_t: Direction similar to $\vec{v}$

a_n: Direction opposite to $\vec{r}$

Vector dot product:

$$\vec{A} \cdot \vec{B} = AB \cos\theta$$

Curvilinear motion of Point *P*



Linear acceleration of Point *P*

$$\vec{a} = \vec{\alpha} \times \vec{r} - \omega^2 \vec{r}$$

$$\therefore \ \vec{a} = a_t \hat{u}_t + a_n \hat{u}_n$$

In scalar format:

$$a_t = \alpha r = \frac{dv}{dt}$$

$$a_n = \omega^2 r = \frac{v^2}{r}$$

Conclusions

- Position vector of a point on a plane is written in $\hat{\imath}$ and $\hat{\jmath}$ unit vector form.
- There are 3 type of rigid body motion: translation, rotation about a fixed axis and general plane motion.
- In the case of rotation, any point on the body undergoes motion in a circular path.
- The direction of instantaneous velocity $\vec{v}$ is always tangent to the curved path.

Planar Kinematics of a Rigid Body (Translation and Rotation)

“Gravity explains the motions of the planets, but it cannot explain who sets the planets in motion.”

– *Sir Isaac Newton*