BFF1303: ELECTRICAL / ELECTRONICS ENGINEERING

Alternating Current Circuits : Sinusoids and Phasors

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Alternating Current Circuit (AC)-Sinusoids and Phasors

BFF1303 ELECTRICAL/ELECTRONICS ENGINEERING

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Contents:

- Outcomes
- Introduction
- Sinusoids
- Phasor
- Phasor Relationship
- Impedance

Outcomes

Draw the power triangle, and compute the capacitor size required to perform power factor correction on a load.

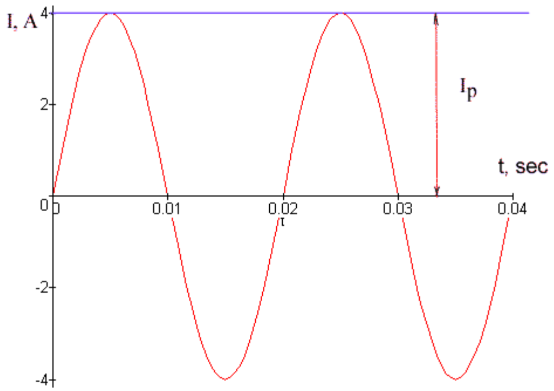
Convert time-domain sinusoidal voltages and currents to phasor notation, and vice versa; represent circuit using impedance

Solve steady-state ac circuits, using phasors and complex impedances.

Learn complex power notation; compute apparent power, real, and reactive power for complex load.

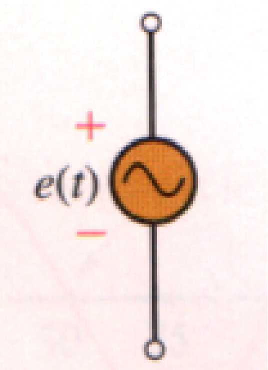
Introduction

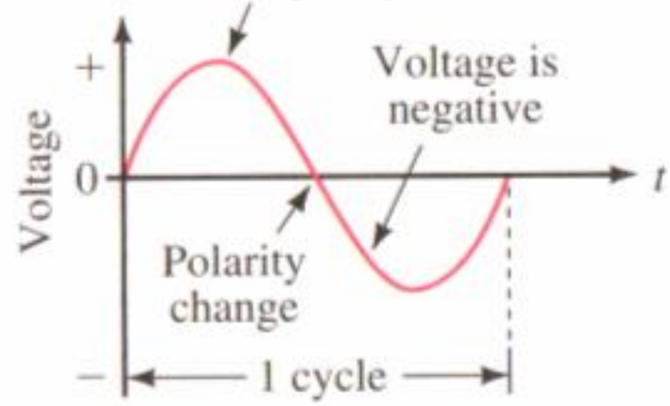
- This chapter will focus on circuit analysis with varying source voltage or current(sinusoidally).
- A sinusoid is a signal that has the form of the sine or cosine function.
- A sinusoidal current is usually referred to as alternating current (AC).



Introduction

- AC is an electrical current whose magnitude and direction vary sinusoidally with time.
- Such a current reverses at regular time intervals and has alternately positive and negative values.





Variation of voltage versus time

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Introduction

- The circuits analysis is considering the time-varying voltage source or current source.

- Circuits driven by sinusoidal current or voltage sources are called ac circuits.

- A sinusoid can be express in either sine or cosine form.

Generating AC Voltage

- One way to generate an ac voltage is to rotate a coil of wire at constant angular velocity in a uniform magnetic field.

- The magnitude of the resulting voltage is proportional to the rate at which flux lines are cut.

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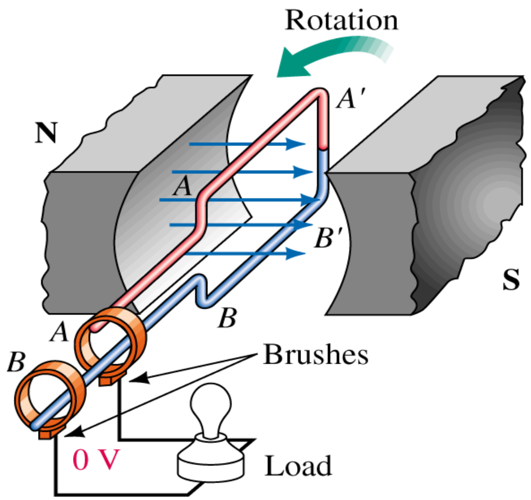
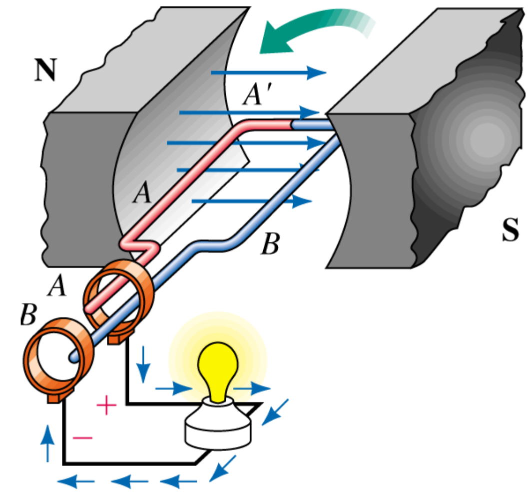
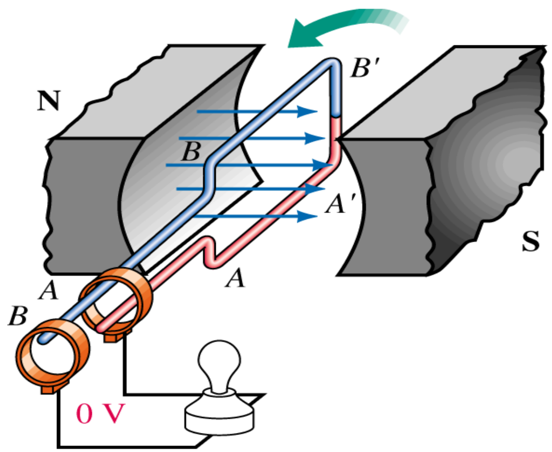
Introduction



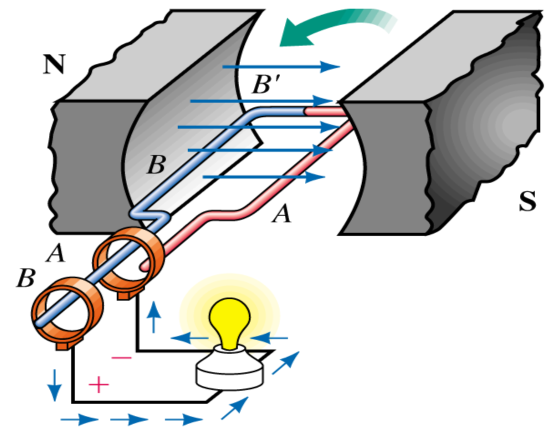
(a) 0° Position: Coil sides move parallel to flux lines. Since no flux is being cut, induced voltage is zero.



(b) 90° Position: Coil end *A* is positive with respect to *B*. Current direction is out of slip ring *A*.



(c) 180° Position: Coil again cutting no flux. Induced voltage is zero.



(d) 270° Position: Voltage polarity has reversed, therefore, current direction reverses.

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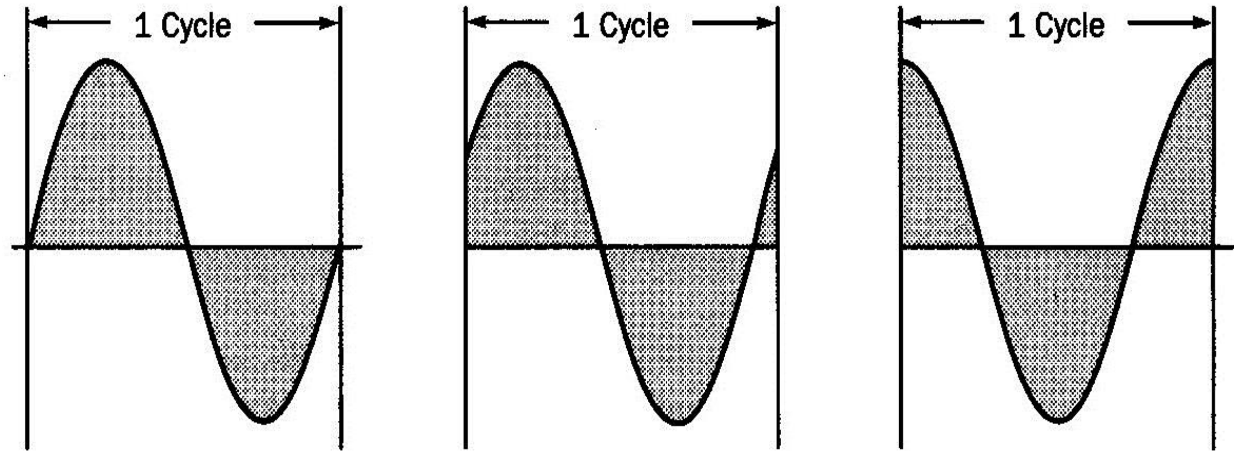
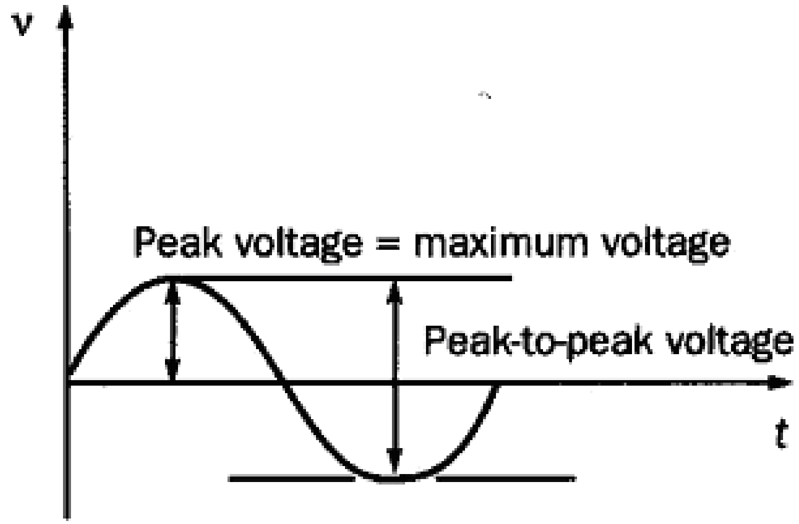
Waveform Definition and Terms

- **Period** – the time taken to complete a cycle, $T(s)$
- **Peak value** – the maximum instantaneous value measured from its zero value, $V_p @ V_m$ (V)
- **Peak-to-peak value** – the maximum variation between the maximum positive instantaneous value and the maximum negative value, V_{p-p} (V)

Introduction





Sinusoids

- Consider the expression of a sinusoidal voltage

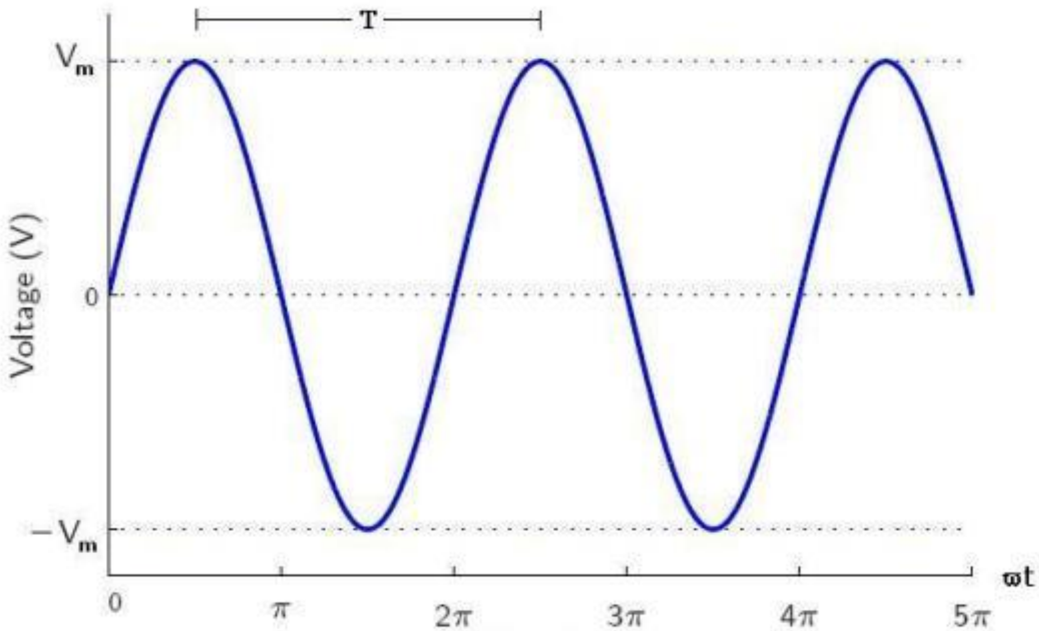
$$v(t) = V_m \sin \omega t$$

angular frequency in radian/second

argument of sinusoid

amplitude of sinusoid

Sinusoids



- The sinusoid repeats itself every T seconds, thus T is called the period of the sinusoid or the time taken to complete one cycle. (s)

Sinusoids

- The number of cycles per second is called frequency, f. (Hz)

$$f = \frac{1}{T}$$

- Angular frequency, ω. (rad/sec)

$$\omega = 2\pi f$$

$$\omega = \frac{2\pi}{T}$$

- An important value of the sinusoidal function is its RMS (root-mean-square) value.

$$V_{rms} = \frac{V_m}{\sqrt{2}} = V_{dc}$$

Sinusoids

ω Usually expressed in radians per second

θ is expressed in degree

$$2\pi \text{ radians} = 360°$$

to convert from degrees to radians, multiply by

$$\frac{\pi}{180}$$

to convert from radians to degrees, multiply by

$$\frac{180}{\pi}$$

Sinusoids

- If the waveform **does not pass** through zero at $t = 0$, it **has a phase shift**.

- For a waveform shifted left,

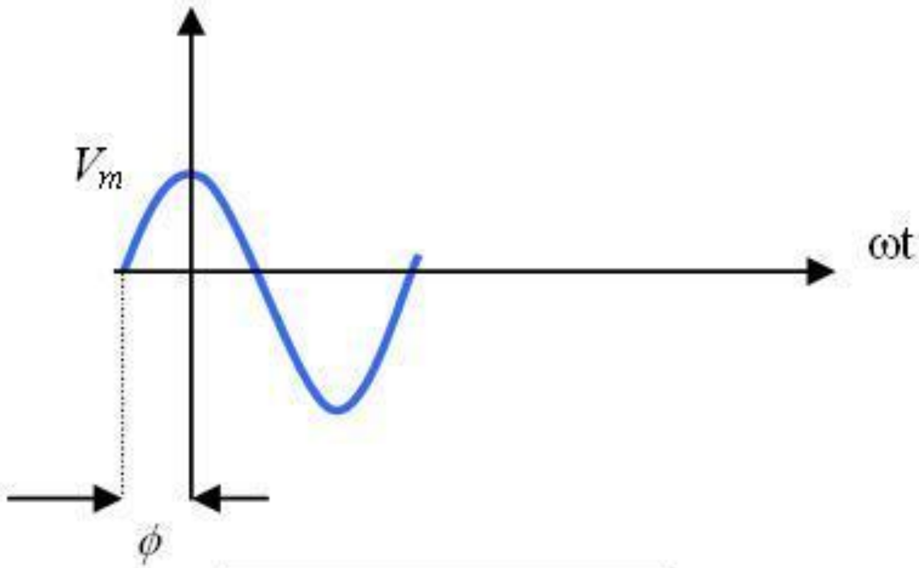
$$v(t) = V_m \sin(\omega + \phi)$$

- For waveform shifted right,

$$v(t) = V_m \sin(\omega - \phi)$$

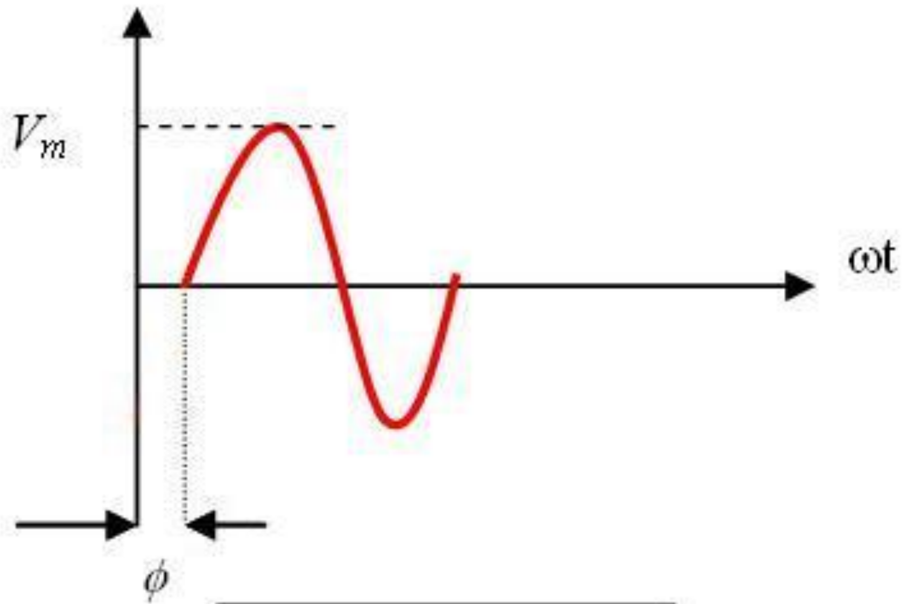
phase angle of sinusoid function

Sinusoids



$$v(t) = V_m \sin(\omega t + \phi)$$

Sinusoids



$$v(t) = V_m \sin(\omega t - \phi)$$

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Example #1

Find the amplitude, phase, period and frequency of the sinusoid

$$v(t) = 12\sin(50t + 10^\circ)$$

Solution

Amplitude $V_m = 12\,\text{V}$

Phase $\phi = 10^\circ$

Angular frequency $\omega = 50\ \text{rad/s}$

Period $T = \dfrac{2\pi}{\omega} = 0.1257\,\text{s}$

Frequency $f = \dfrac{1}{T} = 7.958\,\text{Hz}$

Example #2

A sinusoidal voltage is given by the expression

$$v = 300\cos\left(120\pi t + 30°\right)$$

i. What is the frequency in Hz?
ii. What is the period of the voltage in miliseconds?
iii. What is the magnitude of v at $t = 2.778$ ms?
iv. What is the RMS value of v?

Solution

frequency $$f = \frac{\omega}{2\pi} = 60\,\text{Hz}$$

period $$T = \frac{1}{f} = 16.667\,\text{ms}$$

Example #2

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$$v = 300\cos\left(\left[120\pi \times 2.778\,\mathrm{m}\right] + 30^\circ\right)$$

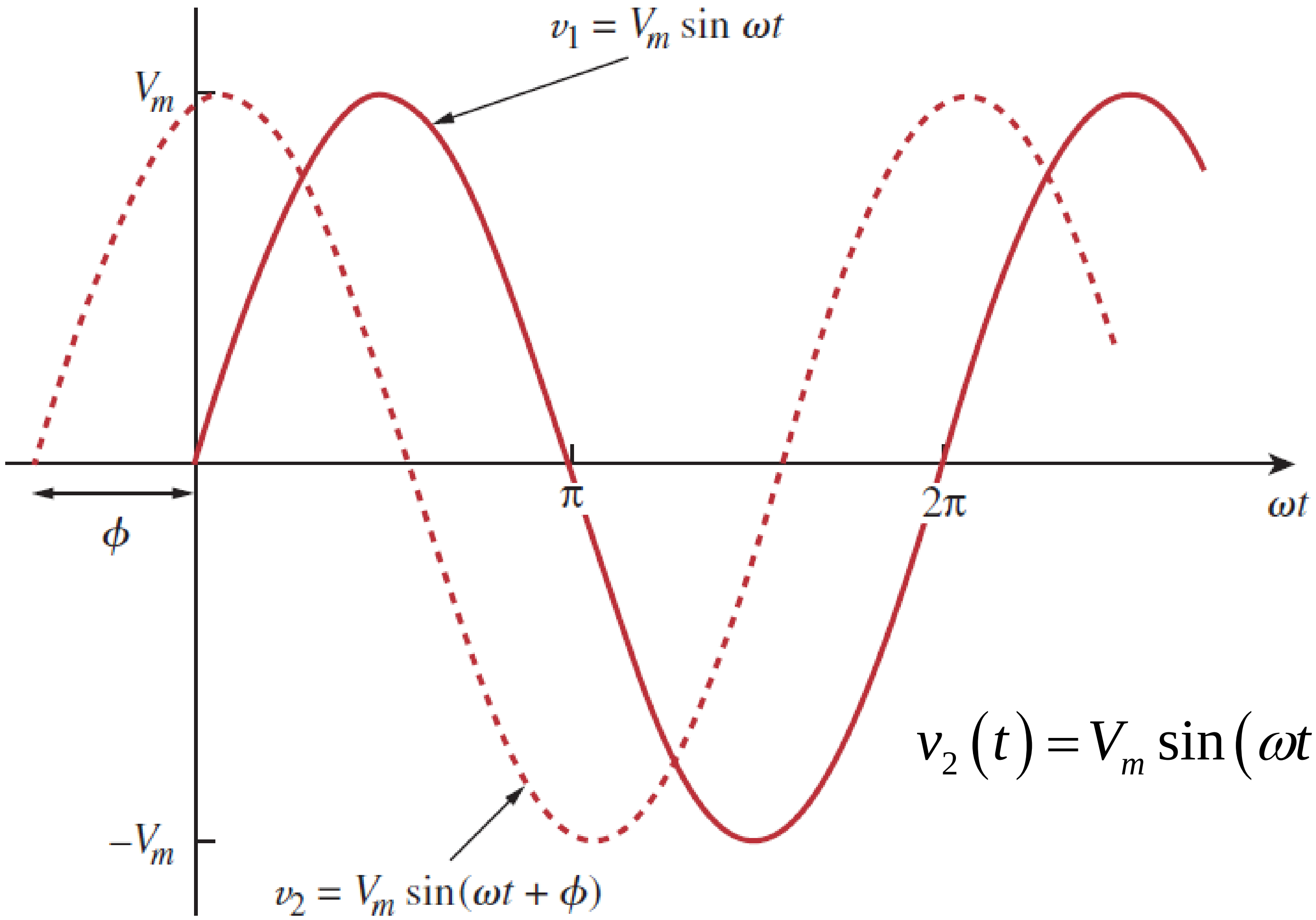
$$v = 300\cos\left(60^\circ + 30^\circ\right) = 0\,\mathrm{V}$$

$$V_{rms} = \frac{V_m}{\sqrt{2}}$$

$$V_{rms} = 212.13\,\mathrm{V}$$

Sinusoids

$$v_1(t) = V_m \sin \omega t$$



$$v_2(t) = V_m \sin(\omega t + \phi)$$

Sinusoids

- The v_2 is occurred first in time.
- Thus it can be said that **v_2 leads v_1** by φ or **v_1 lags v_2** by φ.
- If $\varphi \neq 0$ we can say v_1 and v_2 are **out of phase**.
- If $\varphi = 0$ we can say v_1 and v_2 are **in phase**.
- v_1 and v_2 scan be compared in this manner because they operate at the same frequency (do not need to have the same amplitude).

Sinusoids

- Transformation between cosine and sine form

$$\sin A \Rightarrow \cos(A-90°)$$
$$\cos A \Rightarrow \sin(A+90°)$$

- Converting from negative to positive magnitude

$$-\sin A \Rightarrow \sin(A \pm 180°)$$
$$-\cos A \Rightarrow \cos(A \pm 180°)$$

$$\text{where} \quad A = \omega t + \phi$$

Example #3

For the following sinusoidal voltage, find the value v at $t = 0$ s and $t = 0.5$ s.

$$v = 6\cos\left(100t + 60^\circ\right)$$

Solution

at $t = 0$ s

$$v = 6\cos\left(0 + 60^\circ\right)$$

$$v = 3\,\text{V}$$

at $t = 0.5$ s

$$v = 6\cos\left(50\,\text{rad} + 60^\circ\right)$$

$$v = 4.26\,\text{V}$$

Note: both ωt and φ must be in same unit before adding them up.

Example #4

Calculate the phase angle between

$$v_1 = -10\cos(\omega t + 50°)$$

$$v_2 = 12\sin(\omega t - 10°)$$

State which sinusoid is leading.

Solution

In order to compare v_1 and v_2, we must express them in the same form (either in cosine or sine function) with positive magnitude. Note: the value of φ must be between 0° to ±180°

Example #4

$$v_1 = -10\cos(\omega t + 50°) = 10\cos(\omega t + 50° - 180°)$$

$$v_1 = 10\cos(\omega t - 130°)$$

$$v_2 = 12\sin(\omega t - 10°) = 12\cos(\omega t - 10° - 90°)$$

$$v_2 = 12\cos(\omega t - 100°)$$

the equation v_2 can be written in the following form

$$v_2 = 12\sin(\omega t - 10°) = 12\cos(\omega t - 10° - 90°)$$

$$v_2 = 12\cos(\omega t - 130° + 30°)$$

$' + 30°'$ in the above expression means v_2 leads v_1 by 30°

Phasors

- Sinusoid can be express in terms of phasors, which are more convenient to work with than sine and cosine functions.

- A phasors is a complex number that can be represents the amplitude and the phase of a sinusoid.

- The frequency term is dropped since we know that the frequency of the sinusoidal response is the same as the source.

- The sine/cosine expression is also dropped since we know that the response and source are both sinusoidal.

Phasors

$$r = a \pm jb$$ Rectangular form

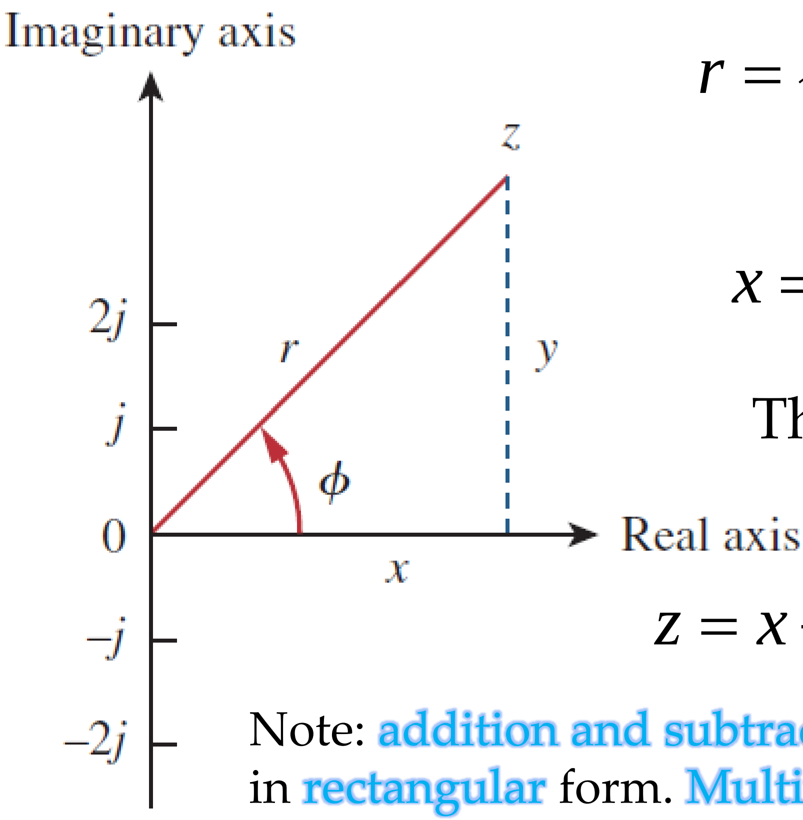
Real part — Imaginary part

$$\mathbf{V} = r\angle\theta$$ Polar form

$$\mathbf{V} = re^{j\theta}$$ Exponential form

Phasors

- The relationship between the rectangular form and the polar form is shown below:



$$r = \sqrt{x^2 + y^2} \qquad \phi = \tan^{-1}\frac{y}{x}$$

$$x = r\cos\phi \qquad y = r\sin\phi$$

Then z can be written as

$$z = x + jy = r\angle\phi = r\cos\phi + j\left(r\sin\phi\right)$$

Note: addition and subtraction of complex number better perform in rectangular form. Multiplication and division in polar form.

Phasors

- Basic properties of complex number:

Addition

$$r_1 + r_2 = (a_1 + a_2) + j(b_1 + b_2)$$

Subtraction

$$r_1 - r_2 = (a_1 - a_2) + j(b_1 - b_2)$$

Multiplication

$$r_1 r_2 = a_1 a_2 \angle(\theta_1 + \theta_2)$$

Division

$$\frac{r_1}{r_2} = \frac{a_1}{a_2} \angle(\theta_1 - \theta_2)$$

Euler Identities

$$\cos(\theta) = \frac{e^{j\theta} - e^{-j\theta}}{2}$$

$$\cos(\theta) = \frac{e^{j\theta} - e^{-j\theta}}{2j}$$

$$e^{j\theta} = \cos(\theta) + j\sin(\theta)$$

$$e^{-j\theta} = \cos(\theta) - j\sin(\theta)$$

$$\left|e^{j\theta}\right| = \sqrt{\cos^2\theta + \sin^2\theta}$$

$$\cos^2\theta + \sin^2\theta = 1$$

$$\left|e^{j\theta}\right| = 1$$

$$\cos\theta = \mathrm{Re}\left(\cos\theta + j\sin\theta\right)$$

Phasors

- First, consider the cosine function as in:

$$v(t) = V_m \cos(\omega t + \phi)$$

- This expression is in time domain.
- In phasor method, we no longer consider in time domain instead in phasor domain (also known as frequency domain).
- The cosine function will be represented in phasor as:

$$\mathbf{V} = V_m \angle \phi$$

Phasors

- The phasor representation carries only the amplitude and phase angle information.
- The frequency term is dropped since we know that the frequency of the sinusoidal response is the same as the source.
- The cosine expression is also dropped since we know that the response and source are both sinusoidal.

Phasors

- Sinusoid-phasor transformation:

$$V_m \cos(\omega t + \phi) \Leftrightarrow V_m \angle \phi$$

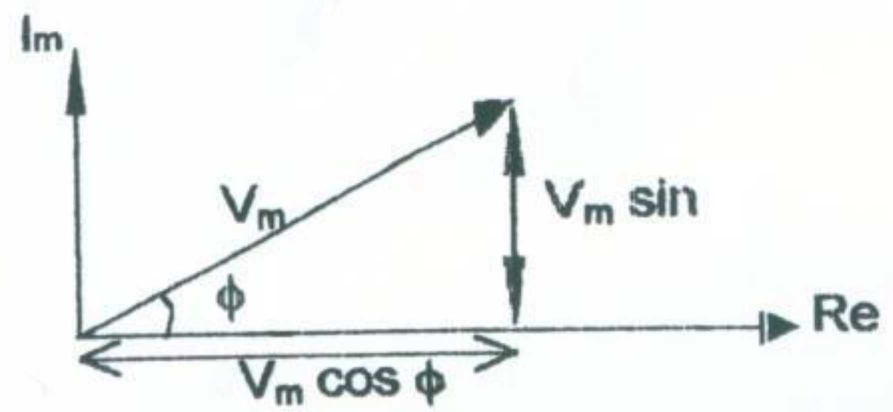
$$V_m \sin(\omega t + \phi) \Leftrightarrow V_m \angle (\phi - 90^\circ)$$

$$I_m \cos(\omega t + \phi) \Leftrightarrow I_m \angle \phi$$

$$I_m \sin(\omega t + \phi) \Leftrightarrow I_m \angle (\phi - 90^\circ)$$

Phasors

- The phasor can be represented by a phasor diagram as shown below.



- Once a sinusoidal voltage or current is represented in its phasor form, simple arithmetic operation can be done.

Example #5

Given $y_1 = 20\cos(100t - 30°)$ and $y_2 = 40\cos(100t + 60°)$. Express $y_1 + y_2$ as a single cosine function.

Solution

In phasor form $y_1 = 20\angle -30°;\ y_2 = 40\angle 60°$

$$y_1 + y_2 = 20\angle -30° + 40\angle 60°$$
$$= (17.31 - j10) + (20 + j34.64)$$
$$= 37.32 + j24.64$$
$$= 44.72\angle 33.4°$$

Thus, $y_1 + y_2 = 44.72\cos(100t + 33.4°)$

Example #6

Suppose that

$$v_1(t) = 20\cos(\omega t - 45)$$

$$v_2(t) = 10\sin(\omega t + 60)$$

Find the total of this voltage and write into polar form

Solution

$$\mathbf{V}_1 = 20\angle -45$$

$$\mathbf{V}_1 = 14.14 - j14.14$$

$$\mathbf{V}_2 = 10\angle -30$$

$$\mathbf{V}_2 = 8.660 - j5$$

Example #6

$$\mathbf{V}_s = \mathbf{V}_1 + \mathbf{V}_2$$

$$\mathbf{V}_s = (14.14 - j14.14) + (8.660 - j5)$$

$$\mathbf{V}_s = 22.80 - j19.14$$

$$\mathbf{V}_s = 29.77\angle -40.01$$

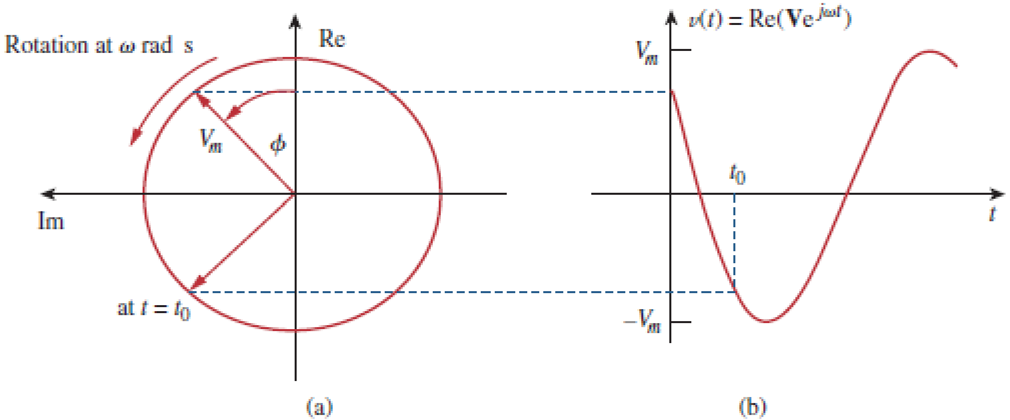
$$v_s(t) = 29.77\cos(\omega t - 40.01)$$

Phasors as rotating vector

- Consider a sinusoidal voltage given by;

$$v(t) = V_m \cos(\omega t + \theta) \qquad v(t) = \mathrm{Re}\left[V_m e^{j(\omega t+\theta)}\right]$$

$$V_m e^{j(\omega t+\theta)} = V_m \angle(\omega t + \theta)$$



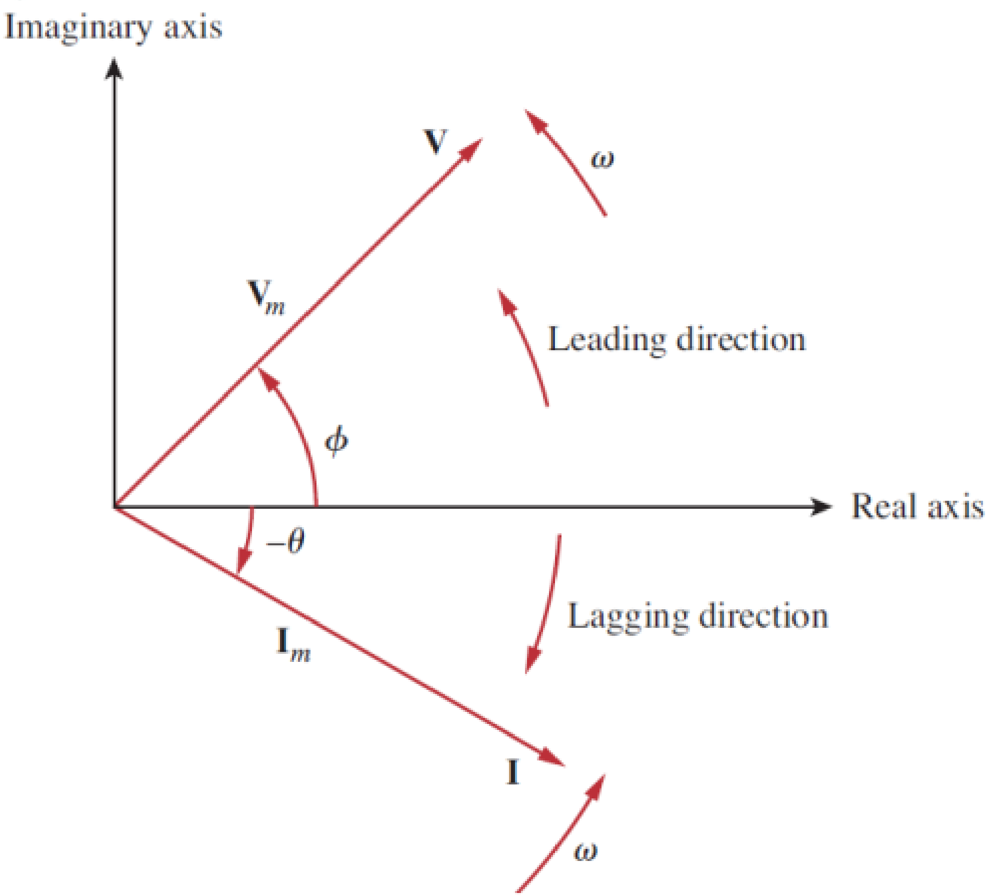
As time increases, the sinor rotates on a circle of radius V_m at angular velocity of ω in the counterclockwise direction

Phasor Relationship

- Consider the phasors

$$\mathbf{V} = V_m \angle \phi$$

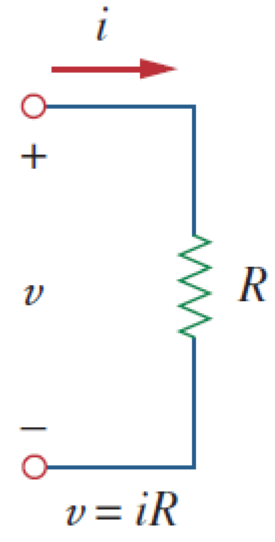
$$\mathbf{I} = I_m \angle -\theta$$



Resistor

- Let the current through the resistor

$$i = I_m \cos(\omega t + \phi)$$



Then the voltage across it is

$$v = iR = RI_m \cos(\omega t + \phi)$$

The phasor form of the voltage

$$\mathbf{V} = RI_m \angle \phi$$

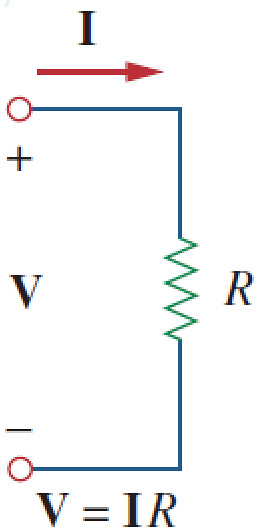
The phasor form of the current

$$\mathbf{I} = I_m \angle \phi$$

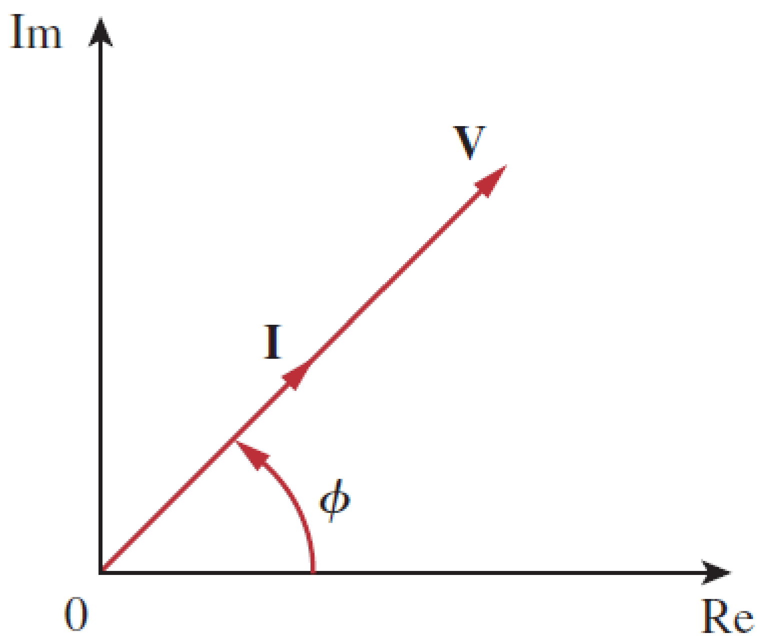
Hence $\mathbf{V} = R\mathbf{I}$

Resistor

$$\mathbf{V} = R\mathbf{I}$$



From the equation, we know that the voltage and current are in phase. Then



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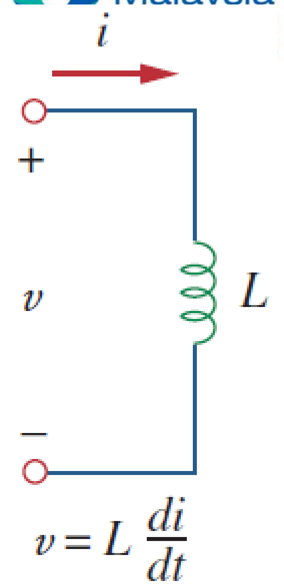
Inductor



- Let the current through the inductor

$$i = I_m \cos(\omega t + \phi)$$

Then the voltage across it is

$$v = L\frac{di}{dt} = -\omega L I_m \sin(\omega t + \phi)$$

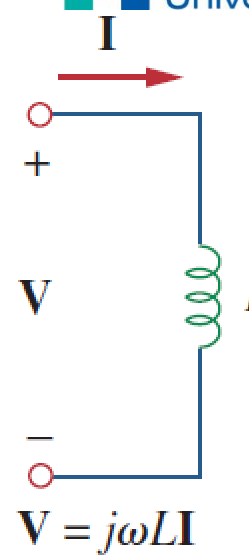
We know that

$$-\sin A = \cos(A + 90°)$$

Then we can write the voltage as

$$v = \omega L I_m \cos(\omega t + \phi + 90°)$$

Inductor



Which transform to phasor

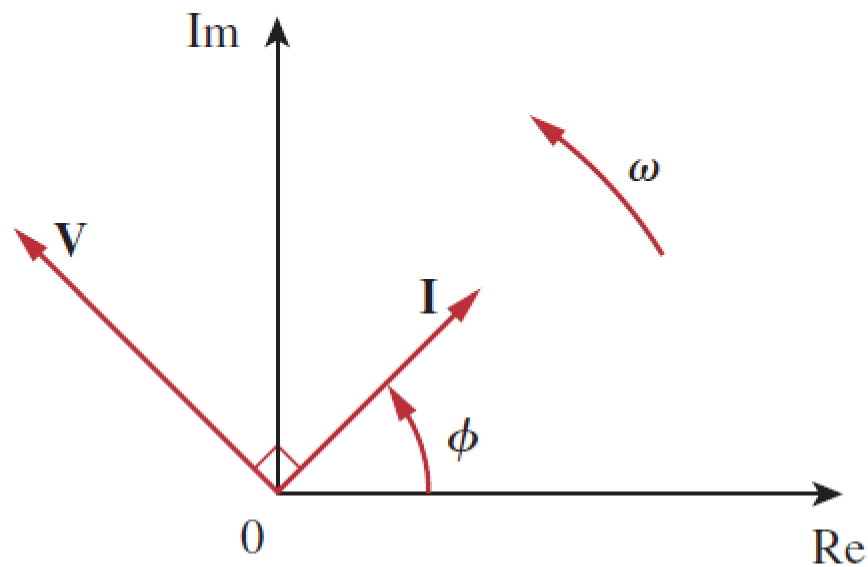
$$\mathbf{V} = \omega L I_m \angle \phi + 90°$$

The phasor form of the current

$$\mathbf{I} = I_m \angle \phi$$

Hence $\mathbf{V} = j\omega L\mathbf{I}$

The voltage and current are 90° out of phase. Specifically, the current lags the voltage by 90°



Capacitor

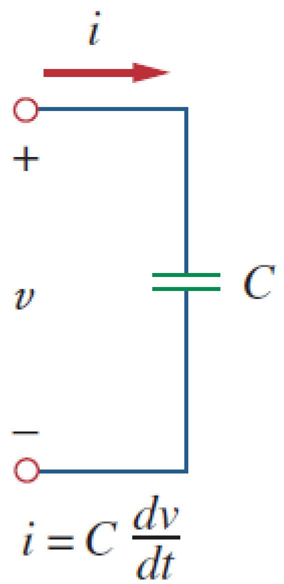
- Let the voltage across the capacitor

$$v = V_m \cos(\omega t + \phi)$$

Then the current through it is

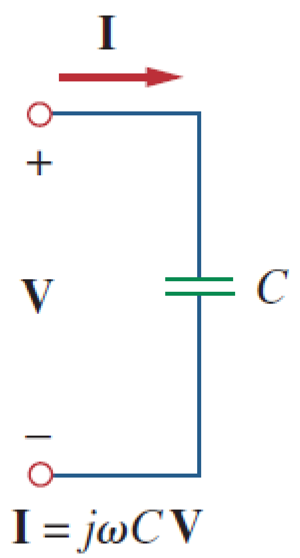
$$i = C\frac{dv}{dt}$$

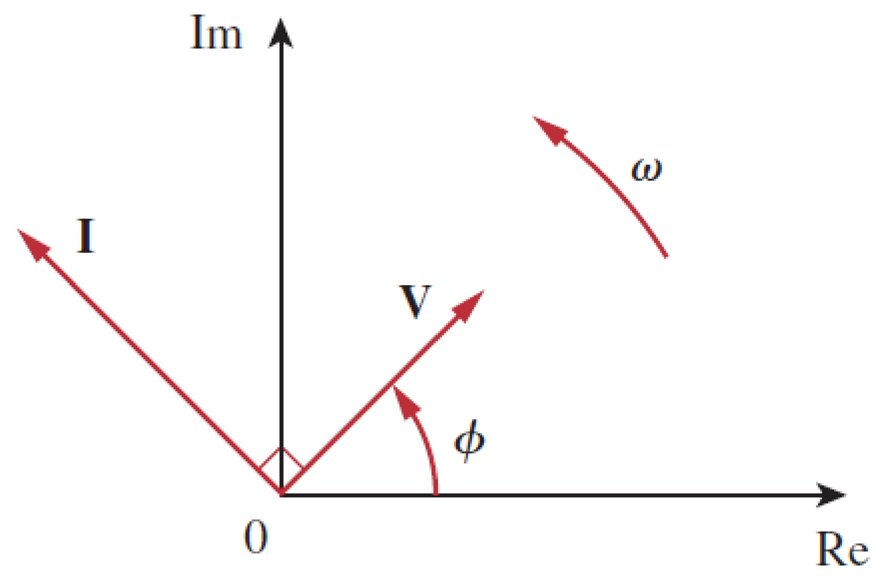
$$\mathbf{I} = j\omega C\mathbf{V} \quad \Rightarrow \quad \mathbf{V} = \frac{\mathbf{I}}{j\omega C}$$



Phasor Relationship

Capacitor





The current and voltage are 90° out of phase. Specifically, the current leads the voltage by 90°

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Summary of voltage-current relationships

Element	Time Domain	Frequency Domain
R	$v = Ri$	$\mathbf{V} = R\mathbf{I}$
L	$v = L\frac{di}{dt}$	$\mathbf{V} = j\omega L\mathbf{I}$
C	$i = C\frac{dv}{dt}$	$\mathbf{V} = \frac{\mathbf{I}}{j\omega C}$

Example #5

The voltage $v = 12\cos(60t + 45°)$ is applied to a 0.1 H inductor. Find the steady-state current through the inductor

Solution

Voltage for inductor is $\mathbf{V} = j\omega L\mathbf{I}$ where $\omega = 60$, $L = 0.1$

$$\mathbf{I} = \frac{\mathbf{V}}{j\omega L} = \frac{12\angle 45}{j60\times 0.1} = \frac{12\angle 45}{6\angle 90}$$

$$\mathbf{I} = 2\angle -45°\ \text{A}$$

$$i(t) = 2\cos(60t - 45°)\ \text{A}$$

Example #6

If voltage $v = 10\cos(100t + 30°)$ is applied to a 50μ F capacitor, calculate the current through the capacitor

Solution

Voltage for capacitor is $\mathbf{V} = \dfrac{\mathbf{I}}{j\omega C}$ where $\omega = 100$, $C = 50\mu$

$$\mathbf{I} = \mathbf{V}(j\omega C) = 10\angle 30(j100 \times 50\mu) = 10\angle 30(5\,\text{m}\angle 90)$$

$$\mathbf{I} = 50\angle 120°\,\text{mA}$$

$$i(t) = 50\cos(100t + 120°)\,\text{mA}$$

Impedance

- The impedance **Z** of a circuit is the ratio of the phasor voltage **V** to the phasor current **I**, measured in ohms (Ω).

Resistor

$$\frac{\mathbf{V}}{\mathbf{I}} = \mathbf{Z}_R$$

Inductor

$$\frac{\mathbf{V}}{\mathbf{I}} = \mathbf{Z}_L$$

Capacitor

$$\frac{\mathbf{V}}{\mathbf{I}} = \mathbf{Z}_C$$

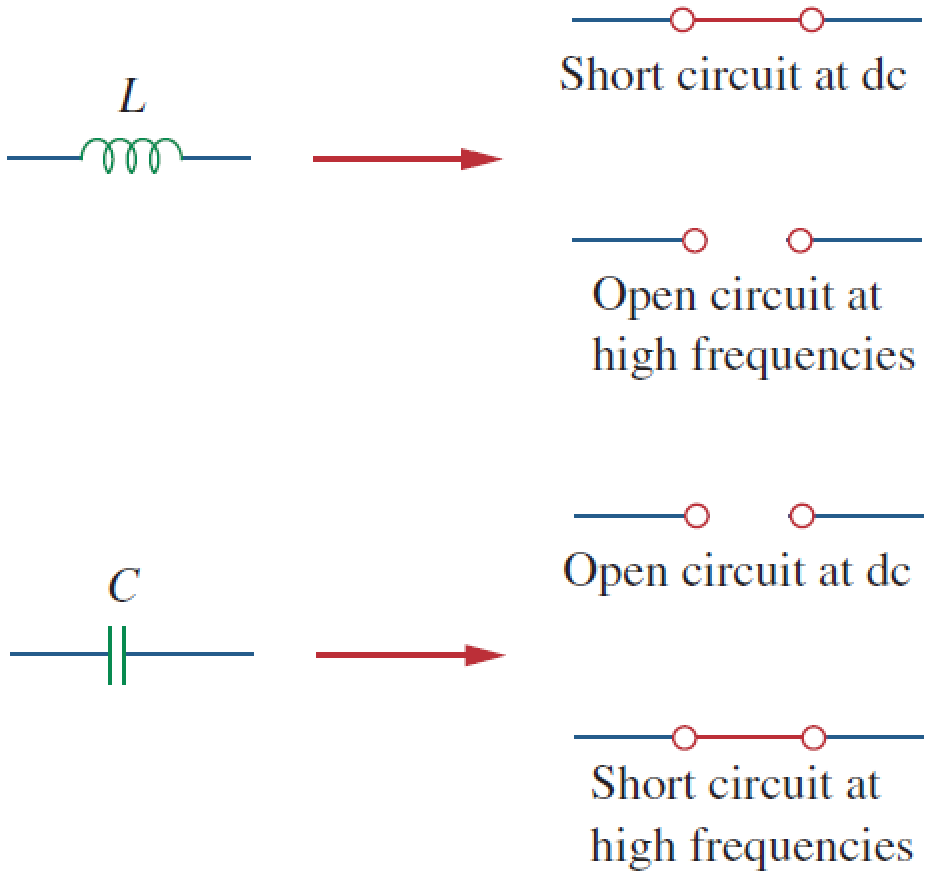
$$\mathbf{Z}_R = R$$

$$\mathbf{Z}_L = j\omega L$$

$$\mathbf{Z}_C = \frac{1}{j\omega C}$$

- When $\omega = 0$, $\mathbf{Z}_L = 0$ inductor short circuit and $\mathbf{Z}_C \rightarrow \infty$ capacitor open circuit
- When $\omega \rightarrow \infty$, $\mathbf{Z}_L \rightarrow \infty$ inductor open circuit and $\mathbf{Z}_C = 0$ capacitor open circuit

Impedance


L
Short circuit at dc
Open circuit at high frequencies
C
Open circuit at dc
Short circuit at high frequencies

Impedance

- As a complex quantity, the impedance may be expressed in rectangular form as

$$\mathbf{Z} = R + jX$$

Real **Z** = resistance

Imaginary **Z** = reactance

- The impedance is inductive when X is positive or capacitive when X is negative.

$\mathbf{Z} = R + jX$ Inductive or lagging since current lags voltage

$\mathbf{Z} = R - jX$ Capacitive or leading since current leads voltage

Impedance

- The impedance may also be expressed in polar as

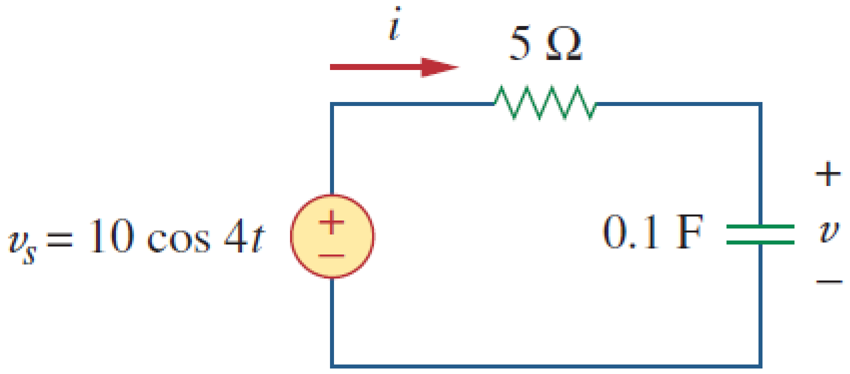
$$\mathbf{Z} = |\mathbf{Z}|\angle\theta$$

$$|\mathbf{Z}| = \sqrt{R^2 + X^2} \qquad \theta = \tan^{-1}\frac{X}{R}$$

$$R = |\mathbf{Z}|\cos\theta \qquad X = |\mathbf{Z}|\sin\theta$$

Example #7

Find $v(t)$ and $i(t)$ in the circuit shown



Solution

$$v_s = 10\cos 4t, \quad \omega = 4$$

$$\mathbf{V}_s = 10\angle 0°$$

The impedance

$$\mathbf{Z} = R + \frac{1}{j\omega C} = 5 + \frac{1}{j(4)(0.1)}$$

$$\mathbf{Z} = 5 - j2.5$$

Example #7

$$\mathbf{V} = \mathbf{IZ}$$

$$\mathbf{I} = \frac{\mathbf{V}}{\mathbf{Z}} = \frac{10\angle 0}{5 - j2.5}$$

$$\mathbf{I} = 1.6 + j0.8 = 1.789\angle 26.57^\circ \text{ A}$$

The voltage across capacitor

$$\mathbf{V} = \mathbf{IZ}_C$$

$$\mathbf{V} = \frac{\mathbf{I}}{j\omega C} = \frac{1.789\angle 26.57^\circ}{5 - j2.5}$$

$$\mathbf{V} = 4.47\angle -63.43^\circ \text{ V}$$

Example #7

In time domain

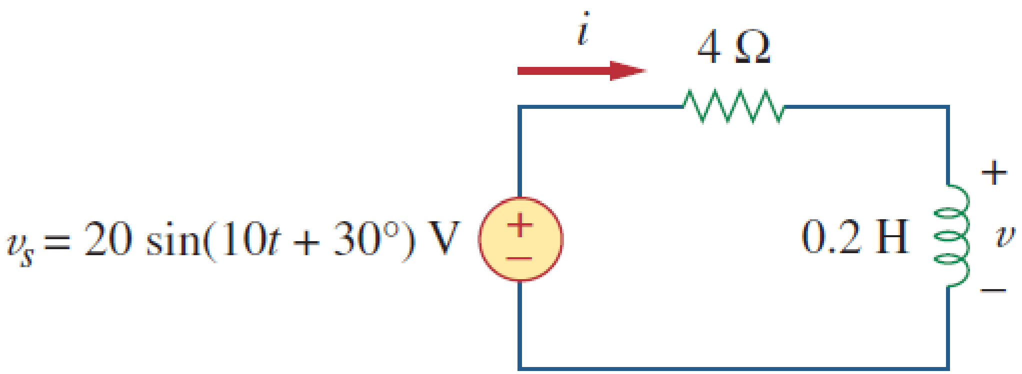
$$i(t) = 1.789\cos(4t + 26.57°)\text{ A}$$

$$v(t) = 4.47\cos(4t - 63.43°)\text{ V}$$

Notice that $i(t)$ leads $v(t)$ by 90° as expected

Example #8

Determine $v(t)$ and $i(t)$



Answer

$$v(t) = 8.944 \sin(10t + 93.43°)\,\text{V}$$

$$i(t) = 4.472 \sin(10t + 3.43°)\,\text{A}$$