# Computational Fluid Dynamics

# Lecture 3

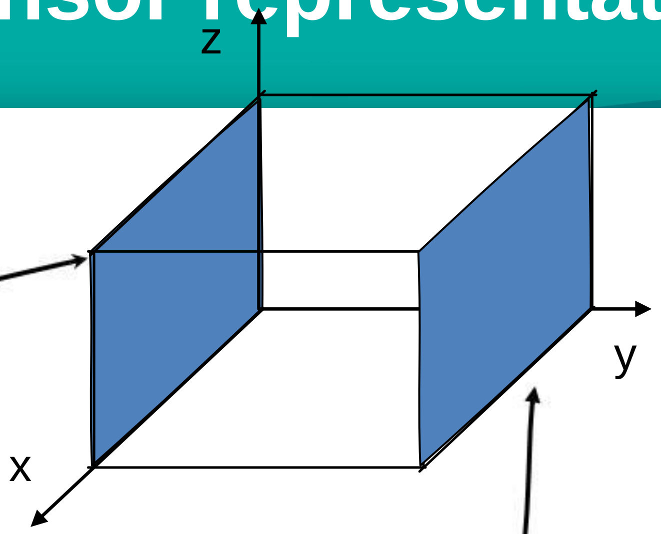
by
Dr. A. Nurye
Faculty of Mechanical Engineering
nurye@ump.edu.my

# Conservation of Momentum and Energy

- Aims
  - The aim of this lecture is to recall the basic momentum equations related to fluid flow and heat transfer.
- Expected Outcomes: At the end of this lecture, students should be able to
  - understand the governing momentum equations
  - derive 2D momentum equations
- References


1) J. Tu, G.H. Yeoh, C. Liu, Computational Fluid Dynamics : A Practical Approach, Elsevier, 1st Edition, 2013.
2) C.T. Shaw, Using Computational Fluid Dynamics, Prentice Hall, 1992

# Tensor representation




Each of the six faces has a **direction**.

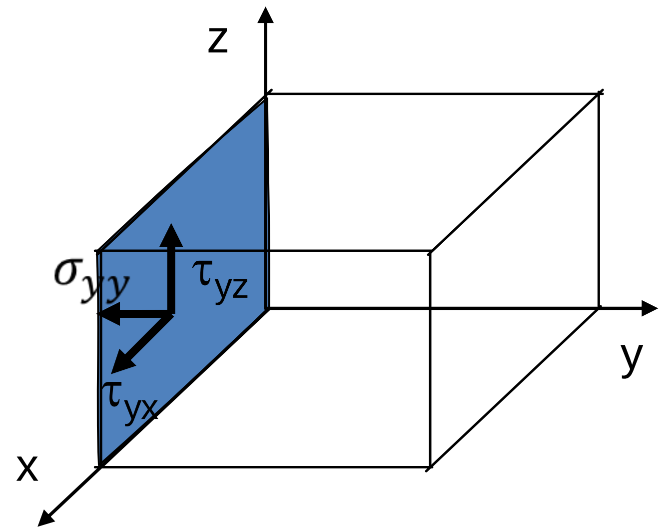
For example, this face

and this face

are normal to the **y direction**

A force acting on any face can act in the x, y and z directions.

Consider the face below.



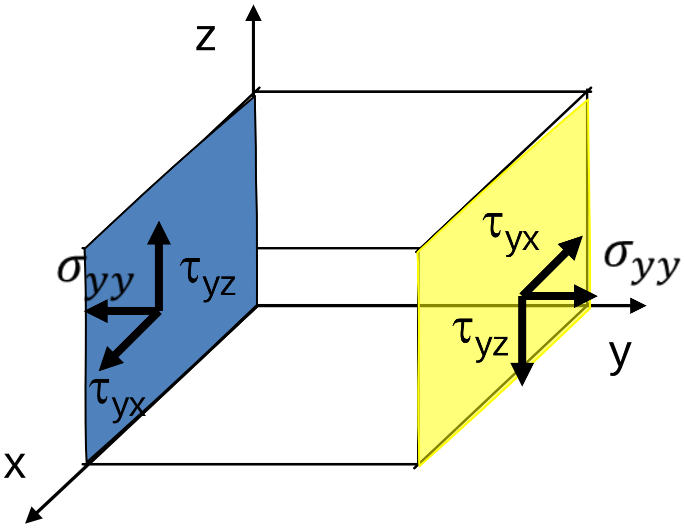

The face is perpendicular to the y direction.

The force per unit face area acting in the x direction on that face is the stress $\tau_{yx}$ (first component: face, second component: stress direction).

The forces per unit face area acting in the y and z directions on that face are the stresses $\tau_{yy}$ and $\tau_{yz}$.

Here $\tau_{yy}$ is a **normal stress** (acts normal, or perpendicular to the face) and $\tau_{yx}$ and $\tau_{yz}$ are **shear stresses** (act parallel to the face)

Some conventions are in order




Normal stresses are defined to be positive **outward**, so the orientation is reversed on the face located $\Delta y$ from the origin

Shear stresses similarly reverse sign on the opposite face.

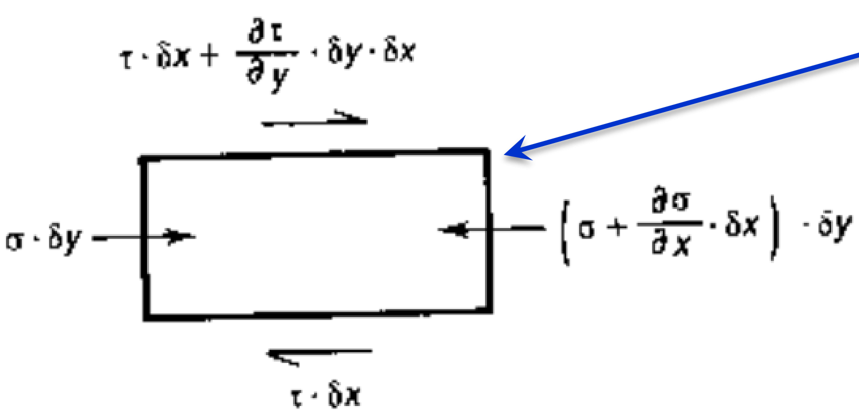
Thus a positive normal stress puts a body in tension, and a negative normal stress puts the body in compression. Shear stresses always put the body in shear.

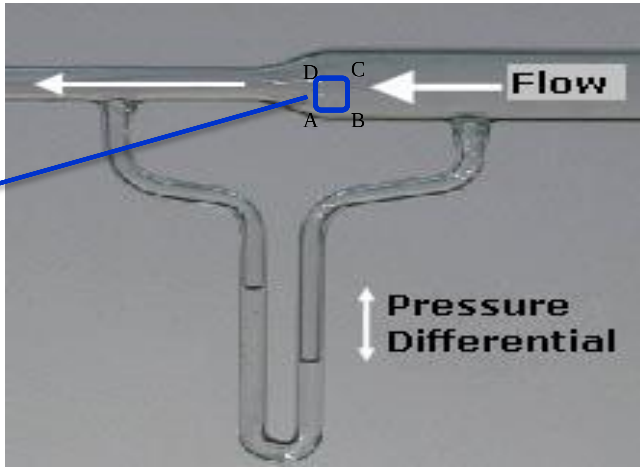
# b) Conservation of momentum

- The basis for the derivation of momentum equations the Newton's $2^{nd}$ Law of Motion

- It is function of the forces on the applied to the fluid and the its acceleration.

| Rate of increase of momentum of fluid particle | = | Sum of forces on fluid particle |
|---|---|---|




https://commons.wikimedia.org/wiki/File:VenturiFlow.png

Consider small portion of the domain fluid as shown above, where $\tau$ is the shear stress (parallel to the surfaces) and $\sigma$ is normal stress

- The acceleration of a fluid particle can be calculated by taking change of velocity with spatial directions ( *x* and *y* ), and time (*t* ).
- Let *u* and *v* be the *x* and *y*, respectively, components of the fluid velocity.
- Thus, the material derivative of *u* will have the form

$$Du = \frac{\partial u}{\partial x} dx + \frac{\partial u}{\partial y} dy + \frac{\partial u}{\partial t} dt$$

- which becomes, on dividing by delta t

$$\frac{du}{dt} = \frac{\partial u}{\partial x}\frac{dx}{dt} + \frac{\partial u}{\partial y}\frac{dy}{dt} + \frac{\partial u}{\partial t}\frac{dt}{dt}$$

- Now, $\frac{dx}{dt}=u$ , and, $\frac{dy}{dt}=v$
- Thus, $\frac{Du}{Dt}=u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}+\frac{\partial u}{\partial t}$
- This expression represents the total acceleration of the fluid in the *x*-direction and is called the material derivative of *u* .
- Newton's Second Law: $F_X=ma_X=m\frac{Du}{Dt}=m\left(u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}+\frac{\partial u}{\partial t}\right)$

where $F_x$, $m$ and $a_x$ are the force, mass, and acceleration of the fluid in the *x*-direction.

- The x-component of the force applied on the fluid element can thus be calculated as

$$\rho\left(u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}+\frac{\partial u}{\partial t}\right)dxdy=\frac{\partial \sigma}{\partial x}dxdy+\frac{\partial \tau}{\partial y}dxdy$$

- But

$$\sigma=-p+2\mu\frac{\partial u}{\partial x},\quad \text{and} \quad \tau=\left(\frac{\partial v}{\partial x}+\frac{\partial u}{\partial y}\right)$$

- Substituting the normal and shear stresses we get the x-momentum equation

$$\rho\left(\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}\right)=-\frac{\partial p}{\partial x}+\mu\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}\right)$$

- Similarly, the y-momentum equation is

$$\rho\left(\frac{\partial v}{\partial t}+u\frac{\partial v}{\partial x}+v\frac{\partial v}{\partial y}\right)=-\frac{\partial p}{\partial y}+\mu\left(\frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 v}{\partial y^2}\right)$$

Where $\mu$ is the viscosity of the fluid and $\rho$ is its density.

## Example

Consider a 2D steady, incompressible viscous flow between two parallel plates a distance *L* apart. Assume that the plates are very wide and very long, so that the flow between the plates is essentially in the x direction. The upper plate moves at velocity *V*. Neglecting the pressure and gravity effect, determine

a) the velocity field,

b) the shear force per unit area acting on the bottom plate,

# Solution

- Step 1: List the given information
  - Assumptions
    1. The plates are infinite in *x* and *z* directions
    2. Steady state flow condition, $\partial/\partial t = 0$
    3. $v=w=0$, the flow is assumed to be parallel
    4. Incompressible flow
    5. No pressure gradient in y-direction
    6. Two dimensional flow, $w=0$, $\partial/\partial z = 0$
    7. No Gravity ,
  - Boundary conditions (Take the origin at the corner of the bottom plate)
    1. At y=0 : $u=0, v=0, w=0$
    2. At y=h : $u=V, v=0, w=0$

## ➢ Step 2: Simplify

Continuity

$$\frac{\partial U}{\partial x} + \frac{\partial V}{\partial y} + \frac{\partial W}{\partial z} = 0$$

$$\frac{\partial U}{\partial x} = 0$$

X-momentum

$$\rho\left(\frac{\partial U}{\partial t} + U\frac{\partial U}{\partial x} + V\frac{\partial U}{\partial y} + W\frac{\partial U}{\partial z}\right) = -\frac{\partial P}{\partial x} + \rho g_x + \mu\left(\frac{\partial^2 U}{\partial x^2} + \frac{\partial^2 U}{\partial y^2} + \frac{\partial^2 U}{\partial z^2}\right)$$

$$\frac{\partial^2 U}{\partial x^2} = 0$$

# ➢Step 3: Integrate

X-momentum

$$\frac{d^2u}{dy^2} = 0 \xrightarrow{\text{integrate}} \frac{du}{dy} = C_1 \xrightarrow{\text{integrate}} \boxed{u(y) = C_1 y + C_2}$$

➢ Step 4: Apply the BC's

- y=0, u=0, $C_1(0) + C_2 \Rightarrow \underline{C_2 = 0}$
- y=h, u=V, $C_1 h \Rightarrow \underline{C_1 = V/h}$

This gives

$$\boxed{u(y) = V\frac{y}{h}}$$

- Step 5: Substitute the velocity values in to the continuity and momentum equations and verify that both equations are conserved.al equations
  - From the x, y, and z, velocity components

$$u = \frac{vy}{h}, \; v = 0, \; and \;\; w = 0$$

  - We get $\frac{\partial u}{\partial x} = 0, \frac{\partial v}{\partial y} = 0, \frac{\partial w}{\partial z} = 0$
  - Continuity is satisfied

$0 + 0 + 0 = 0$

  - X-momentum is satisfied

$$\rho\left(0 + V\frac{y}{h} \cdot 0 + 0 \cdot V/h + 0 \cdot 0\right) = -0 + \rho \cdot 0 + \mu\left(0 + 0 + 0\right)$$

$$0 = 0$$

- Finally, the shear force on bottom plate can be calculated as

$$\tau_{ij} = \begin{pmatrix} 2\mu\frac{\partial U}{\partial x} & \mu\left(\frac{\partial U}{\partial y}+\frac{\partial V}{\partial x}\right) & \mu\left(\frac{\partial U}{\partial z}+\frac{\partial W}{\partial x}\right) \\ \mu\left(\frac{\partial V}{\partial x}+\frac{\partial U}{\partial y}\right) & 2\mu\frac{\partial V}{\partial y} & \mu\left(\frac{\partial V}{\partial z}+\frac{\partial W}{\partial y}\right) \\ \mu\left(\frac{\partial W}{\partial x}+\frac{\partial U}{\partial z}\right) & \mu\left(\frac{\partial W}{\partial y}+\frac{\partial V}{\partial z}\right) & 2\mu\frac{\partial W}{\partial z} \end{pmatrix} = \begin{pmatrix} 0 & \mu\frac{V}{h} & 0 \\ \mu\frac{V}{h} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\frac{\vec{F}}{A} = \tau_w = \mu\frac{V}{h}\hat{i}$$

# CFD EXERCISE

Universiti Malaysia PAHANG
Engineering • Technology • Creativity

Perform a 2D CFD analysis for the laminar incompressible flow of air between two parallel plates discussed in the previous example. The length of the pates is 2 m and the gap between them is 0.2 m. The air is flowing in the horizontal direction along the plate length at velocity of 0.025 m/s.

# Conclusion of the lecture

- The x and y components of a two dimensional conservation of momentum equation

$$\rho\left(\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}\right)=-\frac{\partial p}{\partial x}+\mu\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}\right)$$

$$\rho\left(\frac{\partial v}{\partial t}+u\frac{\partial v}{\partial x}+v\frac{\partial v}{\partial y}\right)=-\frac{\partial p}{\partial y}+\mu\left(\frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 v}{\partial y^2}\right)$$

# Dr. A. Nurye

## Research interest:

- Computational Fluid Dynamics,
- Thermo-fluids,
- Multidisciplinary Numerical Modelling and Simulation

## Contact:

Tel: +094246259
email: nurye@ump.edu.my