Automatic Control

Modelling of Dynamic Systems (Transfer Function: Mechanical Systems)

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Chapter Description

- Aims
  - To expose students to construction of transfer function of several usual systems with different degree of freedom.

- Expected Outcomes
  - Student will be able to create transfer function for translational system
  - Student will be able to create transfer function for rotational system with/without gearing.

- References

Content

- Create transfer function from translational mechanical system
- Create transfer function from rotational mechanical system without gearing
- Create transfer function from rotational mechanical system with gearing

System modelling

In order to control a system, the interested variable need to be observable and manipulated. Therefore, how can we model a system? The steps can be listed as following:

1. Identification of input and output & degree of freedom
2. Identification of constant parameters involved
3. Identification of all forces
4. Transforming the equation into frequency domain and writing down the differential equation
5. Extracting the transfer function

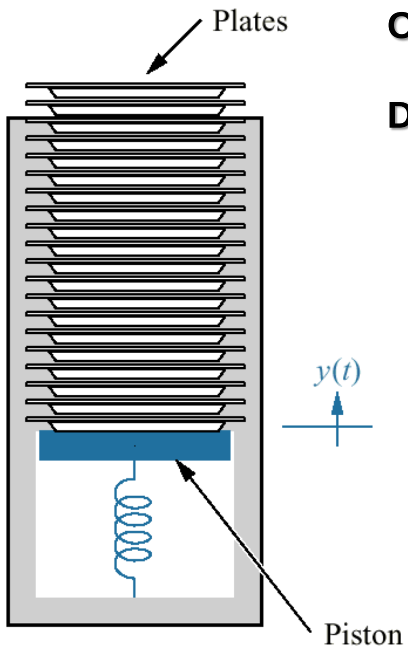
System modeling: Example

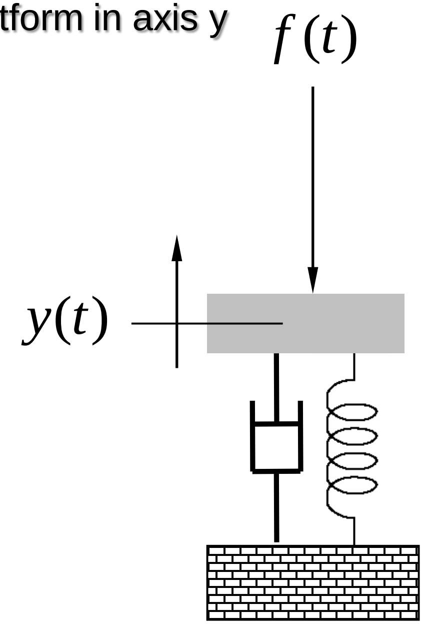
Step 1

Input?: Force generated by the plates mass

Output? : Displacement of the platform in axis y

Degree of freedom? : 1 → y(t)



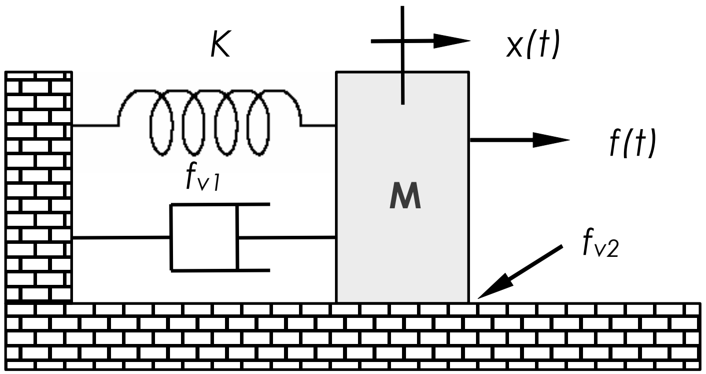


Reference figure: Norman S. Nise, 2008. Control Systems Engineering, sixth Edition, John Wiley & Sons, Inc. (**Chapter 2 – Problem 54**)

System modeling: Example

Step 2

What are the constant parameters involved?
- Spring elasticity, k
- Damper, fv1
- Viscous friction with wall, fv2



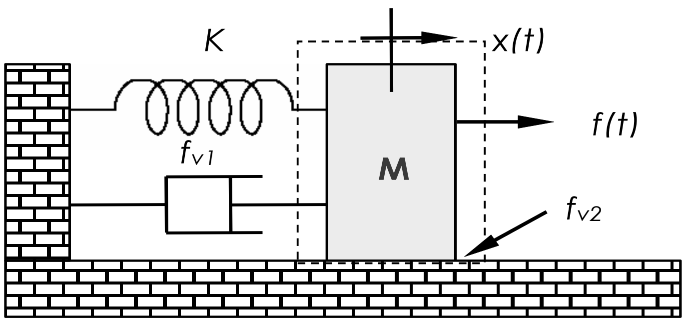
Schematic representation

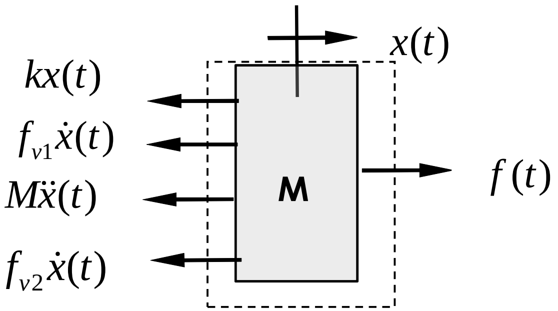
Reference figure: Norman S. Nise, 2008. Control Systems Engineering, sixth Edition, John Wiley & Sons, Inc. (**Chapter 2 – Problem 54**)

System modeling: Example

Step 3

Identify the resulting forces from the parameters found in Step 2 so that we can apply FPD



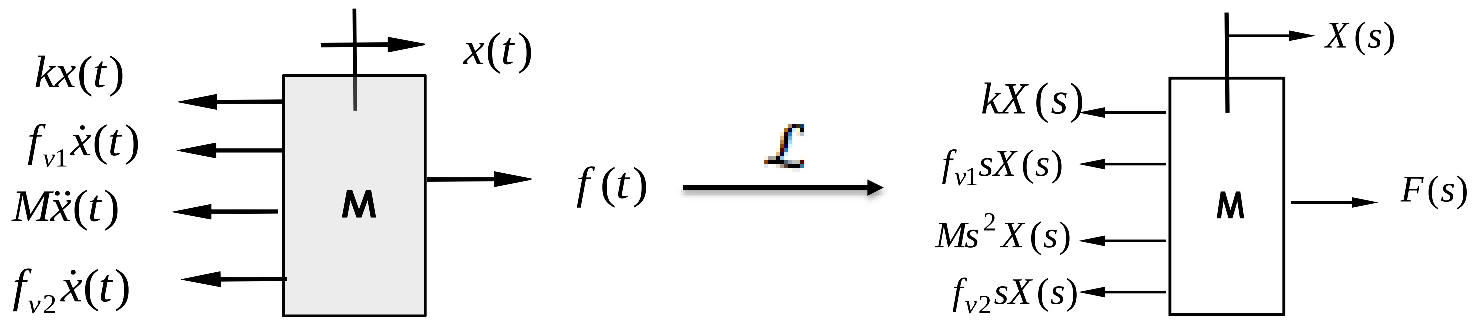


There are five forces including the input force – don't forget the inertial forces

System modeling: Example

Step 4

Transform the time domain equation into frequential domain equation using Laplace transform and write down the differential equation.



$$\underbrace{Ms^2X(s)+(f_{v1}+f_{v2})sX(s)+kX(s)}_{\text{Output}}=\underbrace{F(s)}_{\text{Input}}$$

$$\left(Ms^2+(f_{v1}+f_{v2})s+k\right)X(s)=F(s)$$

System modeling: Example

Step 5

Extract the transfer function that we are looking for

$$Ms^2X(s)+(f_{v1}+f_{v2})sX(s)+kX(s)=F(s)$$

We were interested in the position of the mass $x(t)\rightarrow X(s)$

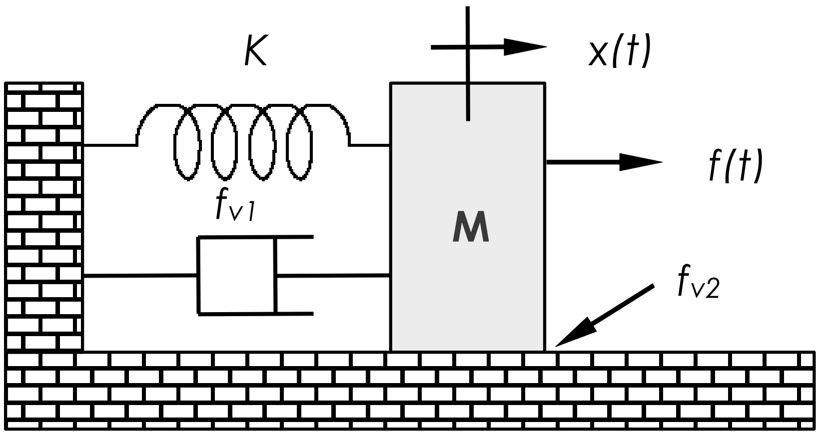
$$\boxed{G(s)=\frac{X(s)}{F(s)}=\frac{1}{Ms^2+(f_{v1}+f_{v2})s+k}}$$

We end with a transfer function G(s) that explain how X(s) varies as consequence of F(s)

System modeling: Example

Remark:

Could there be any other output from the system?
Yes, the derivation of the position, which is the speed and acceleration.



Input variable:

Applied force $f(t)$

Output variable:

Mass position $x(t)$

Mass velocity $\dot{x}(t)$

Mass acceleration $\ddot{x}(t)$

Derived from the mass position

System modeling: Example

Remark:

Could there be any other output from the system?
Yes, the derivation of the position, which is the speed and acceleration.

We had

$$\boxed{G(s) = \frac{X(s)}{F(s)} = \frac{1}{Ms^2 + (f_{v1} + f_{v2})s + k}}$$

If the velocity of the mass is the interested output: $\dot{x}(t) \rightarrow sX(s)$

$$\boxed{G(s)_2 = \frac{X(s)s}{F(s)} = \frac{1}{Ms + (f_{v1} + f_{v2}) + k\frac{1}{s}}}$$

If the acceleration of the mass is the interested output: $\ddot{x}(t) \rightarrow X(s)s^2$

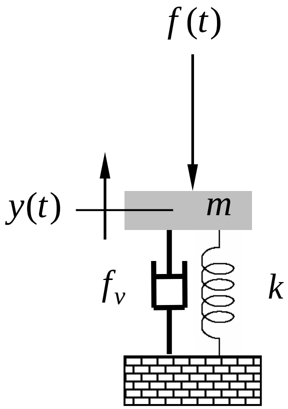
$$\boxed{G(s)_3 = \frac{X(s)s^2}{F(s)} = \frac{1}{M + (f_{v1} + f_{v2})\frac{1}{s} + k\frac{1}{s^2}}}$$

System modeling: Example

Next?

How do we make sense of the transfer function? How can we find the time response when the system is subjected to an input?

As example, for the following values for the parameters;



$$(k + f_v s + ms^2)Y(s) = F(s)$$

$$G(s) = \frac{Y(s)}{F(s)} = \frac{1}{k + f_v s + ms^2}$$

$$k = 2 \text{ N/m}$$

$$f_v = 3 \text{ N.sec/m}$$

$$m = 1 \text{ kg}$$

$$\text{F(t)} = \text{2N}$$

$$\text{F(s)} = 2/\text{s}$$

System modeling: Example

Plug in the parameters value into the transfer function

$$\Rightarrow \quad Y(s) = \frac{2}{s(s+2)(s+1)}$$

Put it in partial fraction form and find the inverse Laplace transform

$$Y(s) = \frac{A}{s} + \frac{B}{s+2} + \frac{C}{s+1}$$

$$A = s\left[\frac{2}{s(s+2)(s+1)}\right]_{s=0} = -1$$

$$B = (s+2)\left[\frac{2}{s(s+2)(s+1)}\right]_{s=-2} = -1$$

$$C = (s+1)\left[\frac{2}{s(s+2)(s+1)}\right]_{s=-1} = 2$$

$$Y(s) = -\frac{1}{s} - \frac{1}{s+2} + \frac{2}{s+1}$$

$$\Big\downarrow \mathcal{L}$$

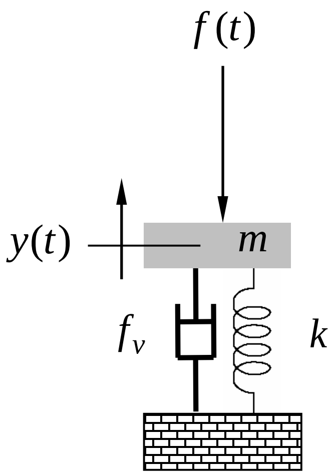
$$y(t) = -1 - 1e^{-2t} + 2e^{-t}$$

$$y(0) = 0$$

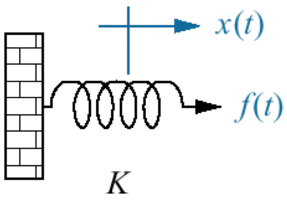
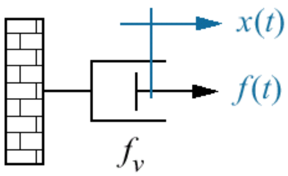
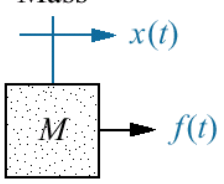
$$y(\infty) = -1$$

System modeling: Example

Plot the time response in time domain

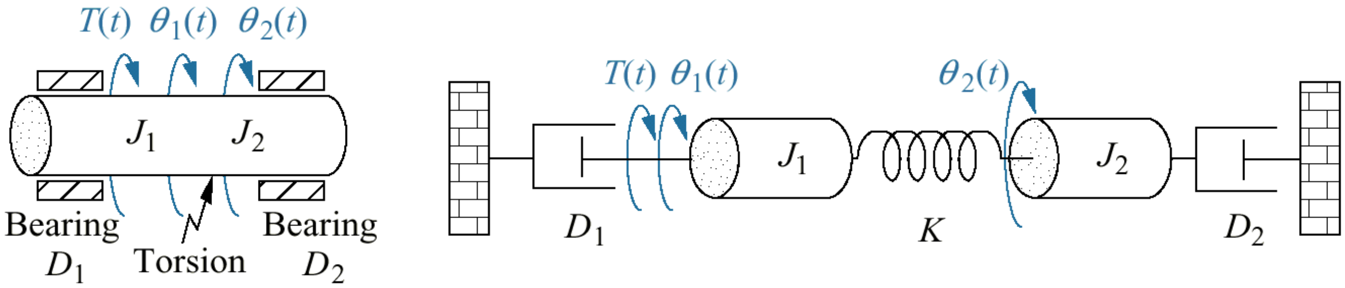


Appendix

Component	Force-velocity	Force-displacement	Impedance $Z_M(s) = F(s)/X(s)$
Spring 	$f(t) = K\int_0^t v(\tau)\,d\tau$	$f(t) = Kx(t)$	K
Viscous damper 	$f(t) = f_v v(t)$	$f(t) = f_v \frac{dx(t)}{dt}$	$f_v s$
Mass 	$f(t) = M\frac{dv(t)}{dt}$	$f(t) = M\frac{d^2x(t)}{dt^2}$	Ms^2

Rotational System

He example shown previously is for a translational system. This part will treat the rotational system.

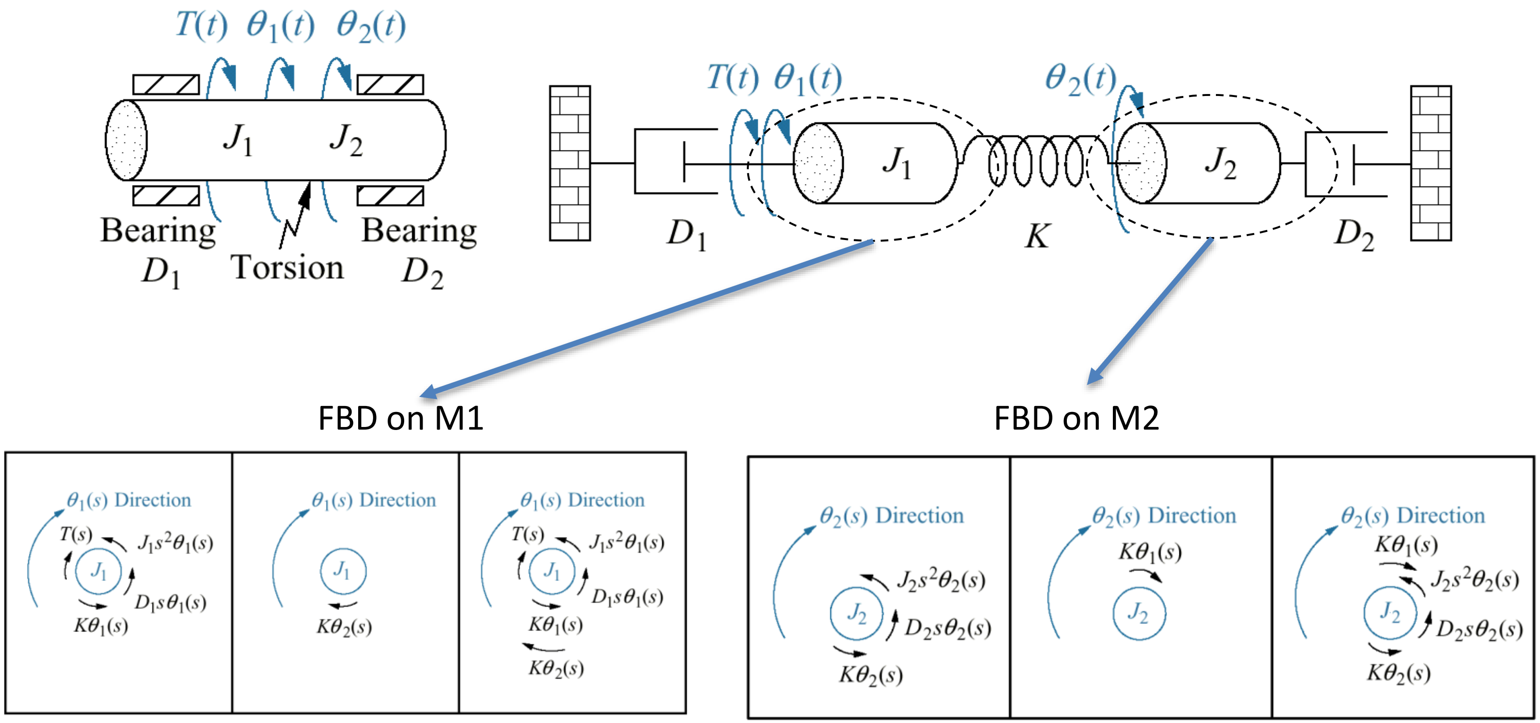


This example is a 2 DoF system. As before, the steps are the same:

1. Identification of input and output & degree of freedom
2. Identification of constant parameters involved
3. Identification of all forces
4. Transforming the equation into frequency domain and writing down the differential equation
5. Extracting the transfer function

Reference figure: Norman S. Nise, 2008. Control Systems Engineering, sixth Edition, John Wiley & Sons, Inc.

Rotational System



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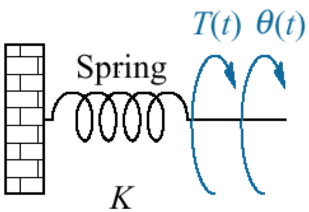
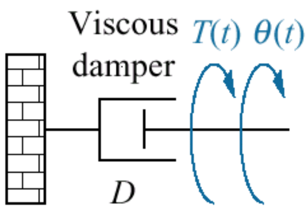
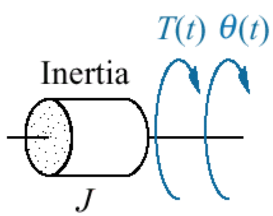
Appendix

Component	Force-velocity	Force-displacement	Impedance $Z_M(s) = F(s)/X(s)$
	$T(t) = K\int_0^t \omega(\tau)\,d\tau$	$T(t) = K\theta(t)$	K
	$T(t) = D\omega(t)$	$T(t) = D\frac{d\theta(t)}{dt}$	Ds
	$T(t) = J\frac{d\omega(t)}{dt}$	$T(t) = J\frac{d^2\theta(t)}{dt^2}$	Js^2

System With Gearing

Basic on gearing system:

Gears increase or reduce angular velocity (while simultaneously decreasing or increasing torque, such that energy is conserved).

Energy of Driving Gear = Energy of Following Gear

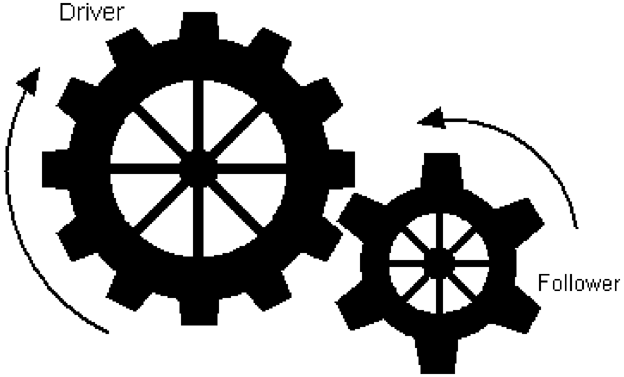
$$N_1\theta_1 = N_2\theta_2$$

$N_1 \longrightarrow$ Number of Teeth of Driving Gear

$\theta_1 \longrightarrow$ Angular Movement of Driving Gear

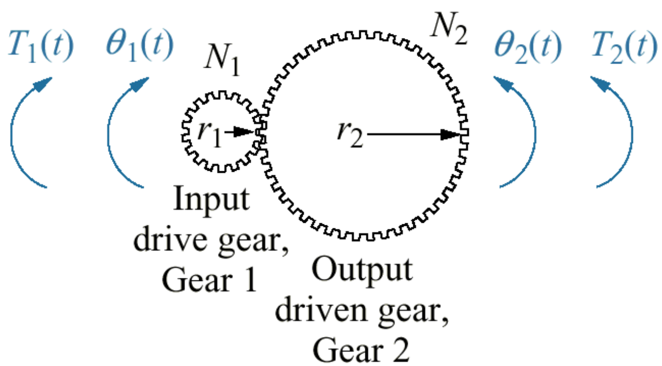
$N_2 \longrightarrow$ Number of Teeth of Following Gear

$\theta_2 \longrightarrow$ Angular Movement of Following Gear

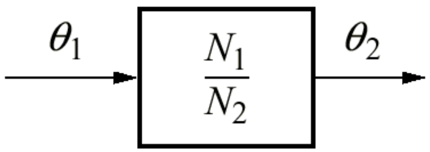


System With Gearing

Basic on gearing system: **Position relationship**



Transfer Function



1. Distance travel (circumference) by Gear 1 must equal distance travel by Gear 2

$$s_1 = s_2$$

$$r_1\theta_1 = r_2\theta_2$$

$$\frac{r_1}{r_2} = \frac{\theta_2}{\theta_1}$$

2. Ratio of radius between Gear 1 and Gear 2 is equal to ratio of number of teeth between Gear 1 and Gear 2

$$\boxed{\frac{N_1}{N_2} = \frac{r_1}{r_2} = \frac{\theta_2}{\theta_1}}$$

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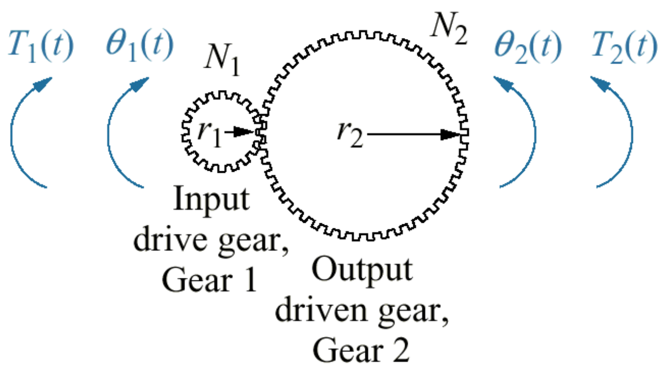
System With Gearing

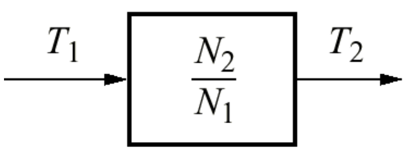
Basic on gearing system: **Torque relationship**



1. Assume work generated by Gear 1 is equal to work consumed by Gear 2

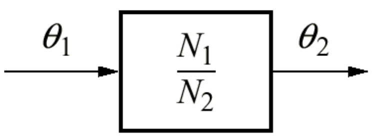
$$W_1 = W_2$$
$$T_1\theta_1 = T_2\theta_2$$

Transfer Function



2. From previous result

$$T_1\theta_1 = T_2\theta_2$$
$$\frac{T_1}{T_2} = \frac{\theta_2}{\theta_1}$$
$$\frac{T_1}{T_2} = \frac{N_1}{N_2}$$



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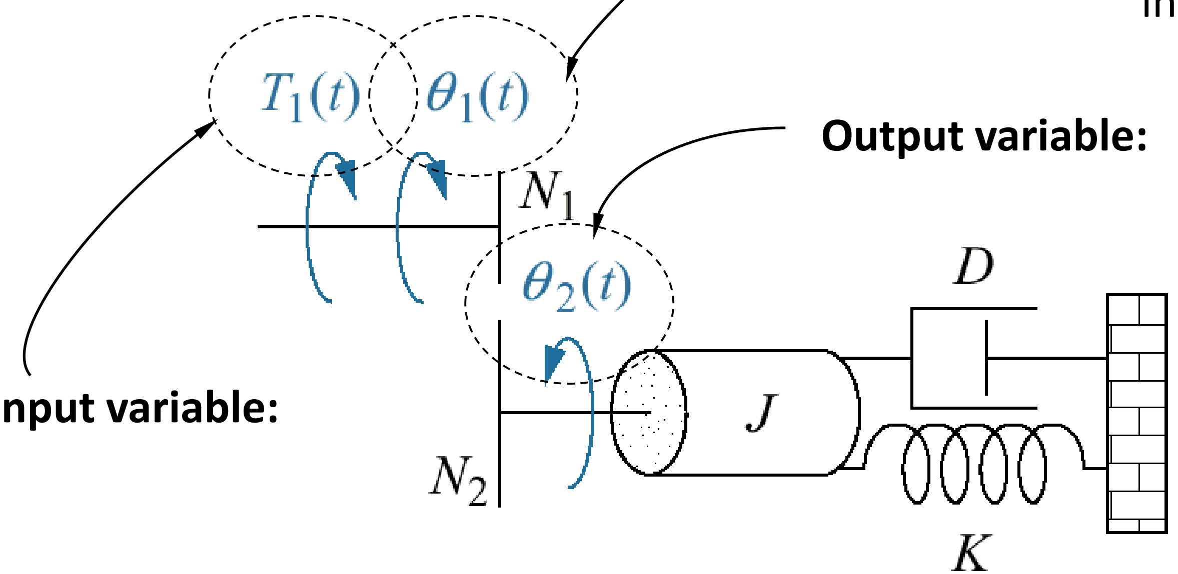
System With Gearing: Example

The steps to construct the transfer function are the same

Output variable: Angular position of input shaft $\theta_1(t)$

Output variable: Angular position of input shaft $\theta_2(t)$

Input variable:

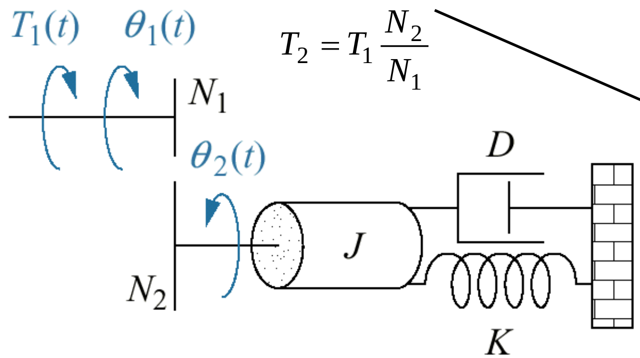


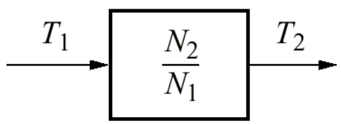
Reference figure: Norman S. Nise, 2008. Control Systems Engineering, sixth Edition, John Wiley & Sons, Inc.

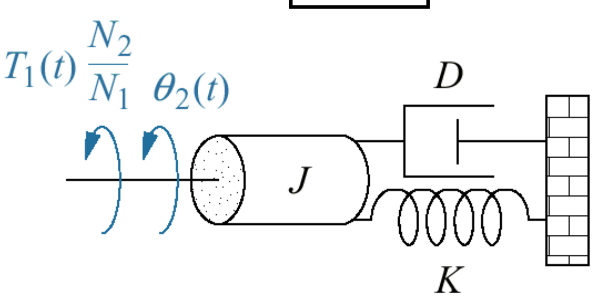
System With Gearing: Example

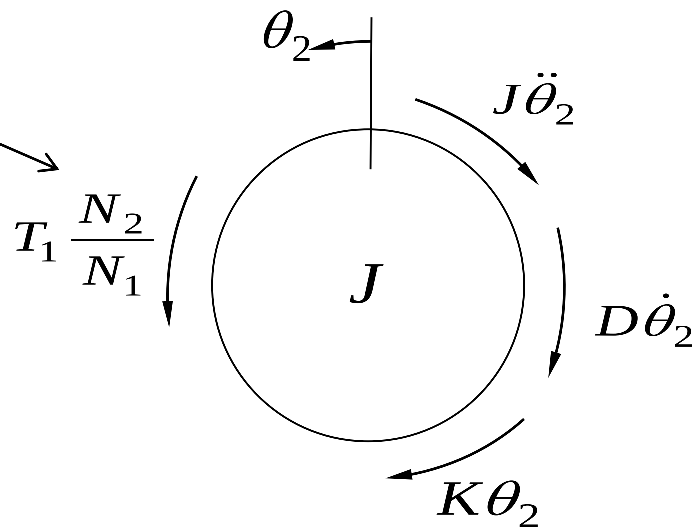
$$\frac{T_1}{T_2} = \frac{N_1}{N_2}$$

$$T_2 = T_1 \frac{N_2}{N_1}$$









$$J\ddot{\theta}_2 + D\dot{\theta}_2 + K\theta_2 = T_1 \frac{N_2}{N_1}$$

$$(Js^2 + Ds + K)\theta_2 = T_1 \frac{N_2}{N_1}$$

System With Gearing: Example

$$J\ddot{\theta}_2 + D\dot{\theta}_2 + K\theta_2 = T_1\frac{N_2}{N_1} \xrightarrow{\mathcal{L}} s^2J\theta_2(s) + sD\theta_2(s) + K\theta_2(s) = T_1\frac{N_2}{N_1}$$

$$\boxed{\frac{\theta_2(s)}{T_1(s)} = \frac{N_2}{N_1(Js^2 + Ds + K)}}$$

What if the we want the transfer function on the first shaft?

Reminder:

$$\theta_1 \longrightarrow \boxed{\frac{N_1}{N_2}} \longrightarrow \theta_2 \qquad \theta_2 = \theta_1\frac{N_1}{N_2}$$

$$(s^2J + sD + K)\theta_2(s) = T_1\frac{N_2}{N_1} \Rightarrow (s^2J + sD + K)\theta_1(s)\frac{N_1}{N_2} = T_1\frac{N_2}{N_1}$$

$$\boxed{\frac{\theta_1(s)}{T_1(s)} = \frac{1}{\left(\left(\frac{N_1}{N_2}\right)^2 Js^2 + \left(\frac{N_1}{N_2}\right)^2 Ds + \left(\frac{N_1}{N_2}\right)^2 K\right)}}$$

Gearing system causes impedance transfer!

$$K_{Destination} = K_{Source}\left[\frac{N_{Destination}}{N_{Source}}\right]^2$$

Thank you

Should there be any question, please contact the author at
mahizami@ump.edu.my

Credit: The slides were developed together with Dr. Gigih Priyandoko of the Faculty of Mechanical Engineering, UMP