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# Automatic Control

# Modelling of Dynamic Systems (Transfer Function)

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# Chapter Description

- Aims
  - To make student understand the interest of using Transfer Function (TF) and familiarize them with TF in studying a system.

- Expected Outcomes
  - Student will be able to explain the difference of time domain and frequency domain
  - Student will be able to manipulate the Laplace transform to construct a TF

- References

# Content

- Understanding time and frequency domain
- Introduction to Transfer Function (TF)
- TF tool : Laplace transform
- Converting differential equation to TF
- Examples & exercises

# Time domain vs. Frequency domain

| Time domain | Frequency domain |
|---|---|
| Observation of variable in function of time | Observation of variable in function of frequency |
| Tracing: Amplitude of variable in function of time | Tracing: Amplitude of variable in function of harmonic ranks |
| Relatively intuitive for any system | Relatively presentation for oscillatory |

- How do we transforms time domain function to frequency domain function?

**LAPLACE TRANSFORM**

$$f(t) \quad \rightarrow \quad F(s)$$

$$L[f(t)] = F(s) = \int_{0-}^{\infty} f(t)e^{-st}dt$$

# Time domain vs. Frequency domain

Table that is used to transform time domain equation to frequency domain equation and vice versa

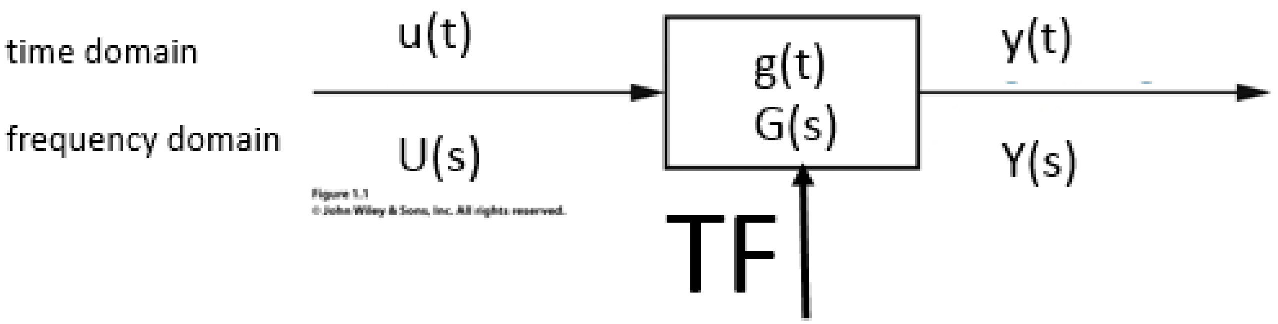
**Table of Laplace Transforms**

| | $f(t)=\mathfrak{L}^{-1}\{F(s)\}$ | $F(s)=\mathfrak{L}\{f(t)\}$ | | $f(t)=\mathfrak{L}^{-1}\{F(s)\}$ | $F(s)=\mathfrak{L}\{f(t)\}$ |
|---|---|---|---|---|---|
| 1. | $1$ | $\frac{1}{s}$ | 2. | $e^{at}$ | $\frac{1}{s-a}$ |
| 3. | $t^n,\quad n=1,2,3,\ldots$ | $\frac{n!}{s^{n+1}}$ | 4. | $t^p, p>-1$ | $\frac{\Gamma(p+1)}{s^{p+1}}$ |
| 5. | $\sqrt{t}$ | $\frac{\sqrt{\pi}}{2s^{\frac{3}{2}}}$ | 6. | $t^{n-\frac{1}{2}},\quad n=1,2,3,\ldots$ | $\frac{1\cdot 3\cdot 5\cdots(2n-1)\sqrt{\pi}}{2^n s^{n+\frac{1}{2}}}$ |
| 7. | $\sin(at)$ | $\frac{a}{s^2+a^2}$ | 8. | $\cos(at)$ | $\frac{s}{s^2+a^2}$ |
| 9. | $t\sin(at)$ | $\frac{2as}{(s^2+a^2)^2}$ | 10. | $t\cos(at)$ | $\frac{s^2-a^2}{(s^2+a^2)^2}$ |
| 11. | $\sin(at)-at\cos(at)$ | $\frac{2a^3}{(s^2+a^2)^2}$ | 12. | $\sin(at)+at\cos(at)$ | $\frac{2as^2}{(s^2+a^2)^2}$ |
| 13. | $\cos(at)-at\sin(at)$ | $\frac{s(s^2-a^2)}{(s^2+a^2)^2}$ | 14. | $\cos(at)+at\sin(at)$ | $\frac{s(s^2+3a^2)}{(s^2+a^2)^2}$ |
| 15. | $\sin(at+b)$ | $\frac{s\sin(b)+a\cos(b)}{s^2+a^2}$ | 16. | $\cos(at+b)$ | $\frac{s\cos(b)-a\sin(b)}{s^2+a^2}$ |
| 17. | $\sinh(at)$ | $\frac{a}{s^2-a^2}$ | 18. | $\cosh(at)$ | $\frac{s}{s^2-a^2}$ |
| 19. | $e^{at}\sin(bt)$ | $\frac{b}{(s-a)^2+b^2}$ | 20. | $e^{at}\cos(bt)$ | $\frac{s-a}{(s-a)^2+b^2}$ |

| | | | | | |
|---|---|---|---|---|---|
| 19. | $e^{at}\sin(bt)$ | $\frac{b}{(s-a)^2+b^2}$ | 20. | $e^{at}\cos(bt)$ | $\frac{s-a}{(s-a)^2+b^2}$ |
| 21. | $e^{at}\sinh(bt)$ | $\frac{b}{(s-a)^2-b^2}$ | 22. | $e^{at}\cosh(bt)$ | $\frac{s-a}{(s-a)^2-b^2}$ |
| 23. | $t^n e^{at},\quad n=1,2,3,\ldots$ | $\frac{n!}{(s-a)^{n+1}}$ | 24. | $f(ct)$ | $\frac{1}{c}F\left(\frac{s}{c}\right)$ |
| 25. | $u_c(t)=u(t-c)$ Heaviside Function | $\frac{e^{-cs}}{s}$ | 26. | $\delta(t-c)$ Dirac Delta Function | $e^{-cs}$ |
| 27. | $u_c(t)f(t-c)$ | $e^{-cs}F(s)$ | 28. | $u_c(t)g(t)$ | $e^{-cs}\mathfrak{L}\{g(t+c)\}$ |
| 29. | $e^{ct}f(t)$ | $F(s-c)$ | 30. | $t^n f(t),\quad n=1,2,3,\ldots$ | $(-1)^n F^{(n)}(s)$ |
| 31. | $\frac{1}{t}f(t)$ | $\int_s^\infty F(u)\,du$ | 32. | $\int_0^t f(v)\,dv$ | $\frac{F(s)}{s}$ |
| 33. | $\int_0^t f(t-\tau)g(\tau)\,d\tau$ | $F(s)G(s)$ | 34. | $f(t+T)=f(t)$ | $\frac{\int_0^T e^{-st}f(t)\,dt}{1-e^{-sT}}$ |
| 35. | $f'(t)$ | $sF(s)-f(0)$ | 36. | $f''(t)$ | $s^2F(s)-sf(0)-f'(0)$ |
| 37. | $f^{(n)}(t)$ | $s^nF(s)-s^{n-1}f(0)-s^{n-2}f'(0)\cdots-sf^{(n-2)}(0)-f^{(n-1)}(0)$ | | | |

# Transfer function (TF)

- What is TF?
  Frequency domain mathematical model that separates input from output




**Y(s) = U(s).G(s)**

Equation in frequency domain, not in time domain !

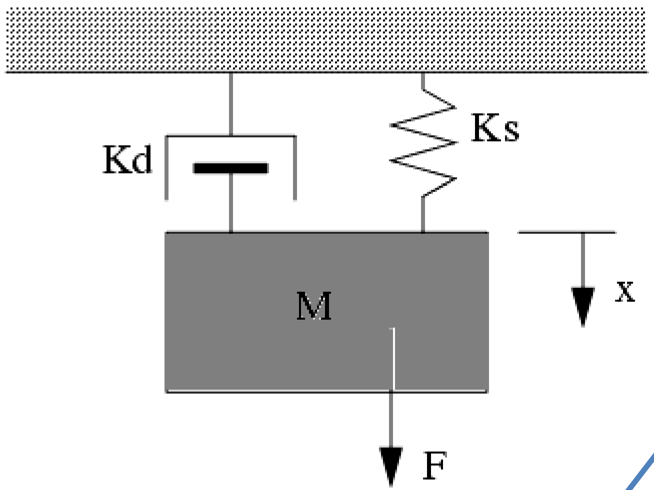
# Transfer function (TF)

- TF can be as simple as a single dynamic like spring-mass system
- It is an easier method to solve differential equation, especially incase of complex differential equation
- A system can be more complex :  example of 3 cascading transfer function
- Simplification is made possible by algebraic operation on transfer functions


$$Y(s) = U(s).G(s).H(s).J(s)$$

# Example of creating transfer function

Consider we want to control/observe the position of the mass M in axis *x, x(t)* in relation to the force *F(t)*




**Use the table in slide pg 5**

Fundamental Principal of Dynamic

$$f = M\ddot{x} + K_d\dot{x} + K_s x$$

Laplace transform into frequency domain

$$\mathscr{L}\{f(t)\} = F(s)$$

$$F(s) = X(s).\ Ms^2 + K_d s + K_s$$

Transfer function = output/input

$$\frac{X(s)}{F(s)} = \frac{1}{Ms^2 + K_d s + K_s}$$

# Mathematical tool: Partial fraction

There will be certain transfer function in a form that is impossible to find its equivalent in the Laplace table. For example,

$$F(s) = \frac{2}{(s+1)(s+2)}$$

In this case, the closest form that can be found in Laplace table is in the form of $\frac{K}{s+p}$
Which is the Laplace transform of $Ke^{-pt}$.

Therefore, $F(s)$ need to be transformed into partial fraction form, Which means finding $K_1$ and $K_2$ for

$$F(s) = \frac{K_1}{s+1} + \frac{K_2}{s+2}$$

$$\Rightarrow \quad L^{-1}[F(s)] = f(t) = K_1 e^{-t} + K_2 e^{-2t}$$

# Mathematical tool: Partial fraction

Example

$$Y(s) = \frac{(s+2)}{s(s+5)} = \frac{A}{s} + \frac{B}{s+5}$$

$$A = \left.\frac{(s+2)}{(s+5)}\right|_{s\to 0} = \frac{2}{5} \qquad B = \left.\frac{(s+2)}{(s)}\right|_{s\to -5} = \frac{3}{5}$$

$$Y(s) = \frac{2/5}{s} + \frac{3/5}{s+5}$$

$$y(t) = \frac{2}{5}e^{0t} + \frac{3}{5}e^{-5t} = \frac{2}{5} + \frac{3}{5}e^{-5t}$$

# Thank you

Should there be any question, please contact the author at
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