BMM3553 Mechanical Vibrations

Chapter 4: Two Degree of Freedom Vibration System (Part 1)

by
Che Ku Eddy Nizwan Bin Che Ku Husin
Faculty of Mechanical Engineering
email: eddy@ump.edu.my

Chapter Description

- Expected Outcomes

  Students will be able to:
  - Develop Equation of Motion (EOM) for Undamped Two-DOF Free and Forced Vibration
  - Determine natural frequencies and mode shape of Undamped Two-DOF Free and Forced Vibration

- References
  - Singiresu S. Rao. Mechanical Vibrations. 5^{th} Ed
  - Abdul Ghaffar Abdul Rahman. BMM3553 Mechanical Vibration Note. UMP.
  - Md Mustafizur Rahman. BMM3553 Mechanical Vibration Lecture Note. UMP

Introduction

- The number of degrees of freedom for a mechanical system can be define as:

No. of degrees of freedom of the system = No. of masses in the system x no. of possible types of motion of each mass

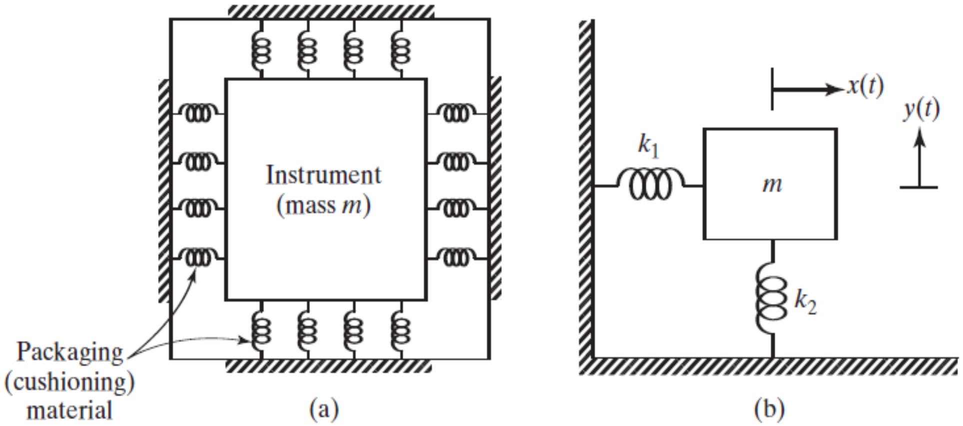
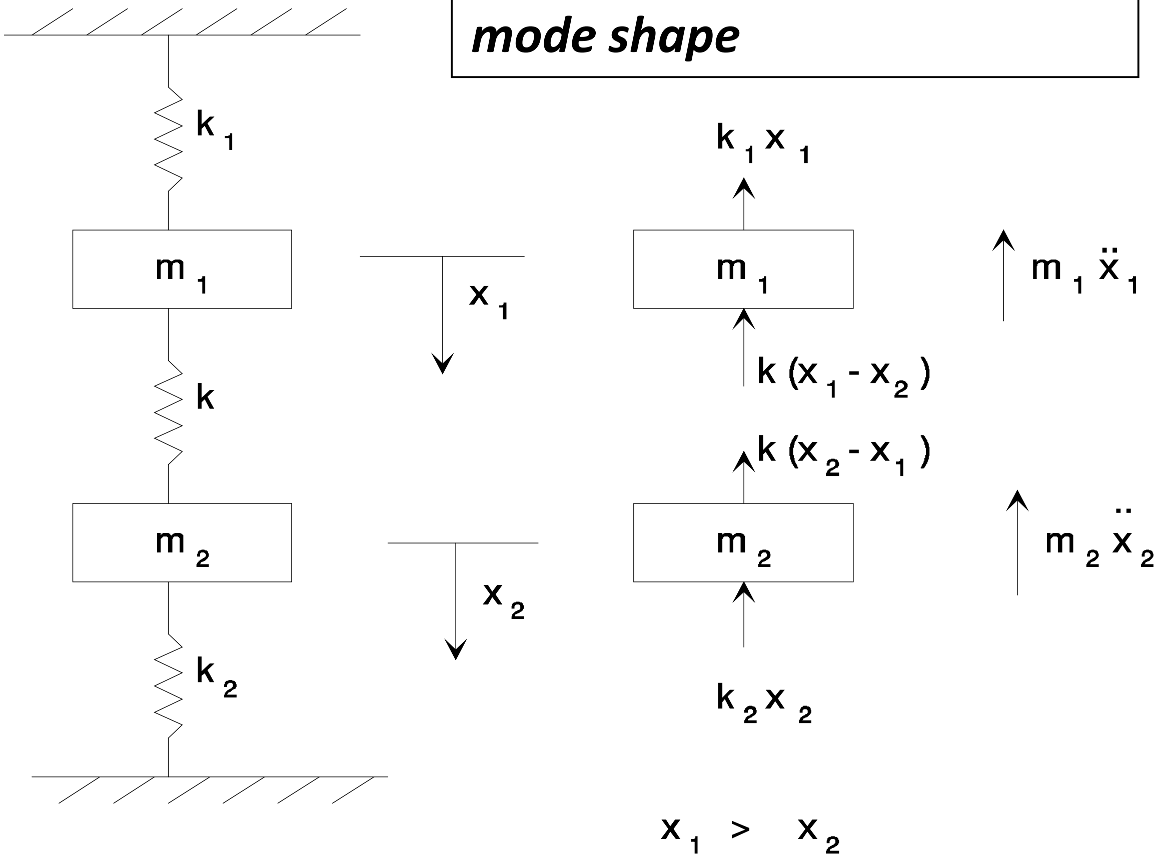


Image source: S.S. Rao 5th Ed

Two Degree of Freedom System

- **Free Vibration without Damping**

Introducing the third dynamic characteristics, namely the ***mode shape***



- <u>**Keywords**</u>
  - natural frequencies
  - mode shapes

Two Degree of Freedom System

- **Mathematical model for 2 DOF system (translation motion)**

$$m_1\ddot{x}_1 + k_1 x_1 + k(x_1 - x_2) = 0$$

$$m_2\ddot{x}_2 + k_2 x_2 + k(x_2 - x_1) = 0$$

$$m_1\ddot{x}_1 + (k_1 + k)x_1 - kx_2 = 0$$

$$m_2\ddot{x}_2 - kx_1 + (k_2 + k)x_2 = 0$$

Two Degree of Freedom System

- Vibrations are usually harmonic in nature and it is reasonable to assume that

$$x_1 = A_1 \sin(\omega t + \phi)$$

$$x_2 = A_2 \sin(\omega t + \phi)$$

$$\ddot{x}_1 = -\omega^2 A_1 \sin(\omega t + \phi)$$

$$\ddot{x}_2 = -\omega^2 A_2 \sin(\omega t + \phi)$$

- Substitute into EOM

$$-m_1\omega^2 A_1 \sin(\omega t + \phi) + (k_1 + k)A_1 \sin(\omega t + \phi) - kA_2 \sin(\omega t + \phi) = 0$$

$$-kA_1 \sin(\omega t + \phi) - m_2\omega^2 A_2 \sin(\omega t + \phi) + (k_2 + k)A_2 \sin(\omega t + \phi) = 0$$

Two Degree of Freedom System

- and if vibration does occur

  sin ($\omega t + \phi$) $\neq$ 0

  Hence,

$$-m_1\omega^2 A_1 + (k_1 + k)A_1 - kA_2 = 0$$

$$-kA_1 - m_2\omega^2 A_2 + (k_2 + k)A_2 = 0$$

$$\begin{bmatrix} -m_1\omega^2 + (k_1 + k) & -k \\ -k & -m_2\omega^2 + (k_2 + k) \end{bmatrix} \begin{bmatrix} A_1 \\ A_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

- Trivial solution $A_1 = A_2 = 0$ gives only static equilibrium condition.

Hence,

$$\begin{vmatrix} -m_1\omega^2 + (k_1 + k) & -k \\ -k & -m_2\omega^2 + (k_2 + k) \end{vmatrix} = 0$$

- This is the characteristic or frequency equation for the system.

Two Degree of Freedom System

- **Expand the equation to determine ω**

$$\left(-m_1\omega^2+(k_1+k)\right)\left(-m_2\omega^2+(k_2+k)\right)-k^2=0$$

$$\omega^4-\left(\frac{k_1+k}{m_1}+\frac{k_2+k}{m_2}\right)\omega^2+\frac{k_1k_2+kk_1+kk_2}{m_1m_2}=0$$

$$\omega^2=\frac{k_1+k}{2m_1}+\frac{k_2+k}{2m_2}\pm\frac{1}{2}\sqrt{\left(\frac{k_1+k}{m_1}+\frac{k_2+k}{m_2}\right)^2-4\frac{k_1k_2+kk_1+kk_2}{m_1m_2}}$$

$$\omega^2=\frac{k_1+k}{2m_1}+\frac{k_2+k}{2m_2}\pm\frac{1}{2}\sqrt{\left(\frac{k_1+k}{m_1}-\frac{k_2+k}{m_2}\right)^2+4\frac{k_1k_2+kk_1+kk_2+kk}{m_1m_2}-4\frac{k_1k_2+kk_1+kk_2}{m_1m_2}}$$

$$\omega^2=\frac{k_1+k}{2m_1}+\frac{k_2+k}{2m_2}\pm\frac{1}{2}\sqrt{\left(\frac{k_1+k}{m_1}-\frac{k_2+k}{m_2}\right)^2+4\frac{kk}{m_1m_2}}$$

$$\omega^2=\frac{k_1+k}{2m_1}+\frac{k_2+k}{2m_2}\pm\sqrt{\frac{1}{4}\left(\frac{k_1+k}{m_1}-\frac{k_2+k}{m_2}\right)^2+\frac{k^2}{m_1m_2}}$$

Two Degree of Freedom System

- Quantities under the square root are positive, hence ω^2 real.
- Four values of ***w:*** $\pm\ w_1$; $\pm\ w_2$
- Negative sign gives no new solution and merely alter the sign of the arbitrary constants.
- The general solution for the D.E. is, therefore,

$$x_1 = A_{11}\sin(\omega_1 t + \phi_1) + A_{12}\sin(\omega_2 t + \phi_2)$$
$$x_2 = A_{21}\sin(\omega_1 t + \phi_1) + A_{22}\sin(\omega_2 t + \phi_2)$$

$$\dot{x}_1 = A_{11}\omega_1\cos(\omega_1 t + \phi_1) + A_{12}\omega_2\cos(\omega_2 t + \phi_2)$$
$$\dot{x}_2 = A_{21}\omega_1\cos(\omega_1 t + \phi_1) + A_{22}\omega_2\cos(\omega_2 t + \phi_2)$$

- Final solution for 2dof :

$$x_1 = A_{11}\cos\omega_1 t + A_{12}\cos\omega_2 t$$
$$x_2 = A_{21}\cos\omega_1 t + A_{22}\cos\omega_2 t$$

Two Degree of Freedom System

where, the natural frequencies,

w_1 = first natural frequency

w_2 = second natural frequency

A_{ij} = amplitude of mass i in mode j

Two Degree of Freedom System

- The amplitudes and phases are determined by the initial conditions.
- The amplitude ratios are important and are obtained from x_1 and x_2

$$\frac{A_{11}}{A_{21}} = \frac{k}{(k_1 + k) - m_1\omega_1^2} = \frac{(k_2 + k) - m_2\omega_1^2}{k}$$

$$\frac{A_{12}}{A_{22}} = \frac{k}{(k_1 + k) - m_1\omega_2^2} = \frac{(k_2 + k) - m_2\omega_2^2}{k}$$

EXERCISE: QUESTION 3 [30 Marks] _ Final Exam 2014/15 (II)

Figure Q3a shows a schematic diagram an overhead crane. The cabin is at the center of the beam of the length, l_1. Reduce the system to an equivalent two degree of freedom system as shown in **Figure Q3b.** Noted that stiffness k_1 is given by the expression as simply supported beam and k_2 express as two cables are subjected to axial loads. Given EI = 21 MNm^2, m_1 = 3000 kg, l_1=5 m, EA = 82.5 MN, m_2 = 700 kg, l_2=6 m. Determine:

a) Stiffness for k_1 and k_2. **(4 Marks)**

b) Develop that the equation of motion can be written in matrix form as

$$\begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix}\begin{bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{bmatrix}+\begin{bmatrix} k_1+2k_2 & -2k_2 \\ -2k_2 & 2k_2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}=\begin{bmatrix} 0 \\ 0 \end{bmatrix}$$ **(8 Marks)**

c) Natural frequencies of the system. **(10 Marks)**

d) The principle mode shapes of the system which normalized to the first mass. Sketch the mode shapes. **(8 Marks)**

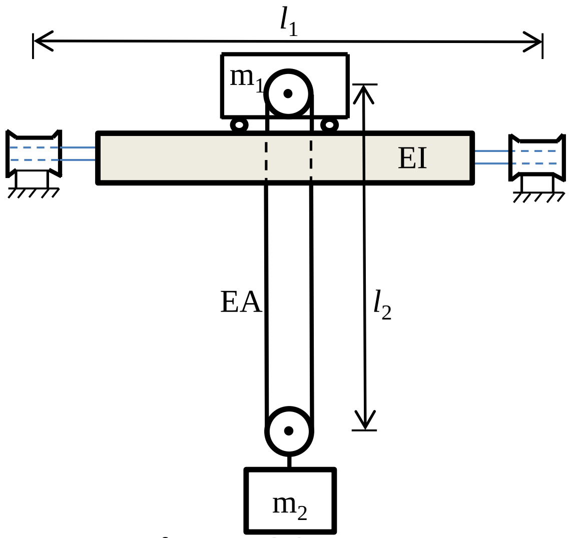


Figure Q3a

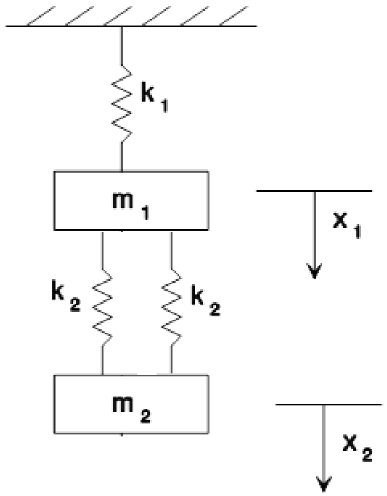


Figure Q3b

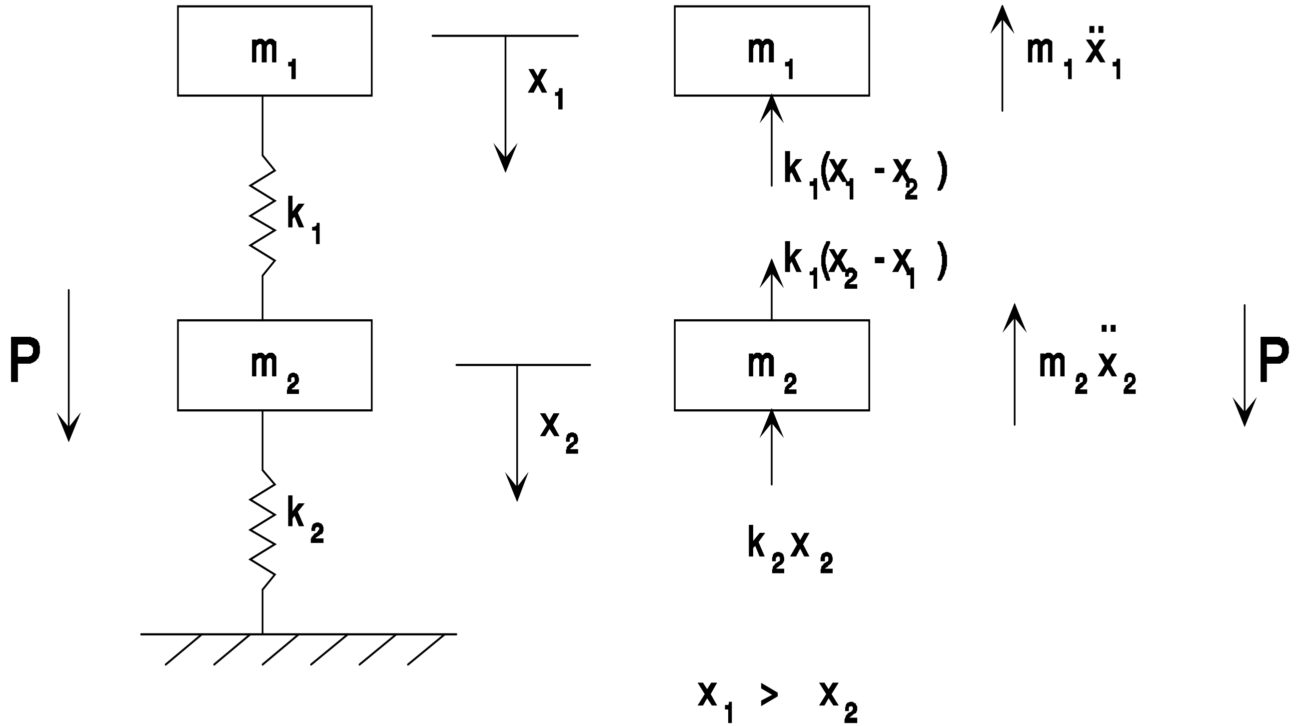
$$for\ fixed - fixed\ beam\ with\ load\ at\ the\ middle, \quad k_{eq} = \frac{192EI}{l^3}$$

$$for\ simply\ supported\ beam\ with\ load\ at\ the\ middle, \quad k_{eq} = \frac{48EI}{l^3}$$

$$Rod\ under\ axial\ loading\ , \quad k_{eq} = \frac{AE}{l}$$

Two Degree of Freedom : Forced Vibration

- It has been seen that a **2-degree of freedom** system has **two natural frequencies**. It is to be expected therefore that when forced the system will have **two resonant frequencies**.



Two Degree of Freedom : Forced Vibration

- The equations of motion are:

$$m_1\ddot{x}_1 + k_1(x_1 - x_2) = 0 \qquad (1)$$

$$m_2\ddot{x}_2 + k_2x_2 + k_1(x_2 - x_1) = P \qquad (2)$$

- Let $\boldsymbol{P = P_o sinwt}$. The steady state vibration of m_2 must be in phase with P and that of m_1 must either be in phase or 180^o out of phase. Therefore, let

$$x_1 = A\sin(\omega t)$$

$$x_2 = B\sin(\omega t)$$

- The phase angle will be decided by the signs of A and B.

Two Degree of Freedom : Forced Vibration

- Substitute into EOM gives

$$[k_1 - m_1\omega^2]A\sin\omega t - k_1 B\sin\omega t = 0 \qquad (1a)$$

$$[-m_2\omega^2 + (k_1 + k_2)]B\sin\omega t - k_1 A\sin\omega t = P_0 \sin\omega t \qquad (2a)$$

$$\begin{bmatrix} k_1 - m_1\omega^2 & -k_1 \\ -k_1 & -m_2\omega^2 + (k_1 + k_2) \end{bmatrix} \begin{bmatrix} A \\ \mathbf{B} \end{bmatrix} = \begin{bmatrix} 0 \\ P_o \end{bmatrix}$$

- The natural frequencies for the system is when RHS =0. Hence, determinant

$$\begin{vmatrix} k_1 - m_1\omega^2 & -k_1 \\ -k_1 & -m_2\omega^2 + (k_1 + k_2) \end{vmatrix} = 0$$

Two Degree of Freedom : Forced Vibration

- Using Cramer's Rule

$$\begin{bmatrix} a_1 & b_1 \\ a_2 & b_2 \end{bmatrix}\begin{bmatrix} A \\ B \end{bmatrix} = \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$$

$$A = \frac{\begin{vmatrix} c_1 & b_1 \\ c_2 & b_2 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}}$$

$$B = \frac{\begin{vmatrix} a_1 & c_1 \\ a_2 & c_2 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}}$$

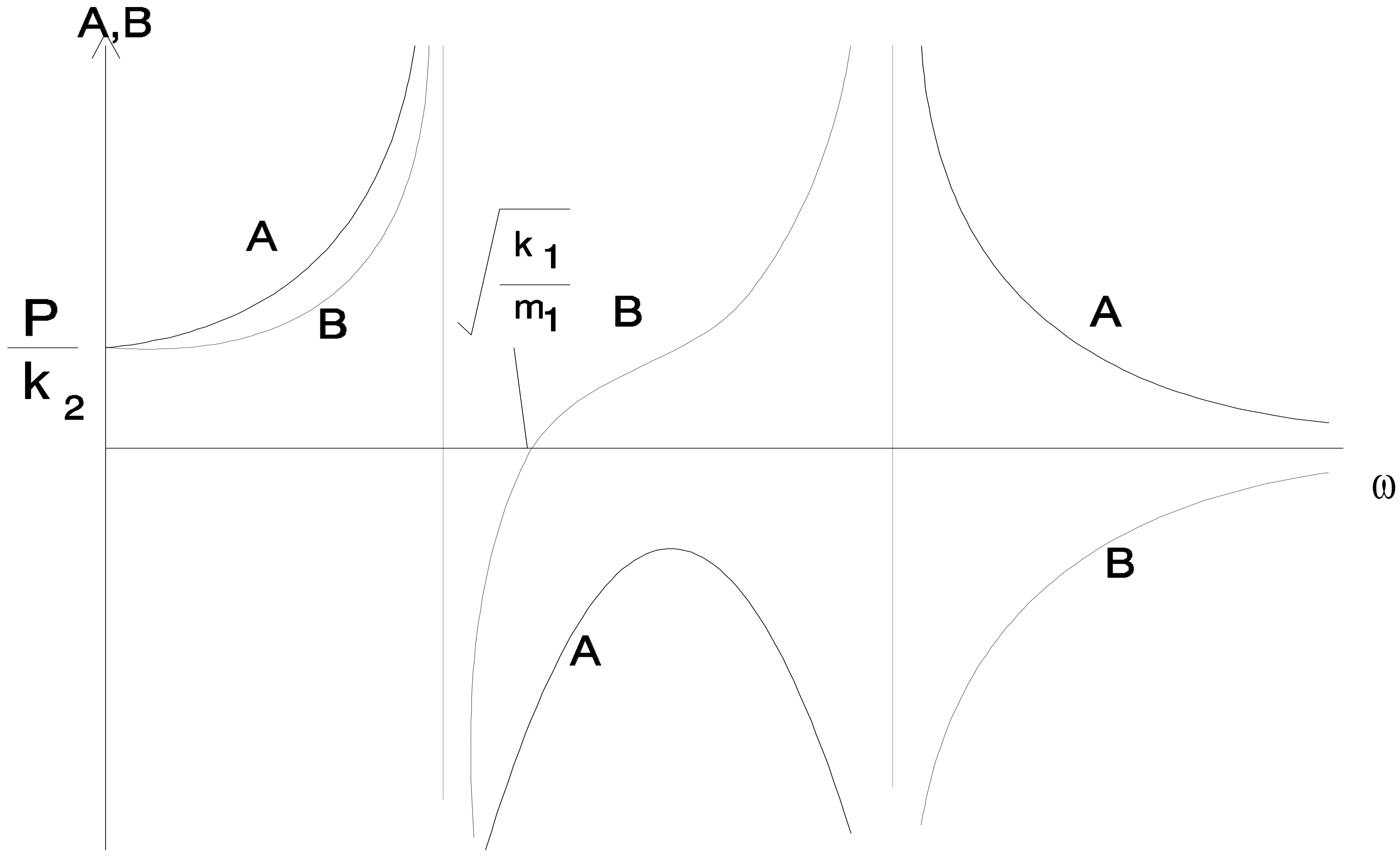
For A

$$\frac{\begin{vmatrix} 0 & -k_1 \\ P_o & -m_2\omega^2 + (k_1 + k_2) \end{vmatrix}}{\begin{vmatrix} k_1 - m_1\omega^2 & -k_1 \\ -k_1 & -m_2\omega^2 + (k_1 + k_2) \end{vmatrix}} = \frac{k_1 P_o}{[-m_2\omega^2 + (k_1 + k_2)][k_1 - m_1\omega^2] - k_1^2}$$

For B

$$\frac{\begin{vmatrix} k_1 - m_1\omega^2 & 0 \\ -k_1 & P_o \end{vmatrix}}{\begin{vmatrix} k_1 - m_1\omega^2 & -k_1 \\ -k_1 & -m_2\omega^2 + (k_1 + k_2) \end{vmatrix}} = \frac{\left(k_1 - m_1\omega^2\right) P_o}{[-m_2\omega^2 + (k_1 + k_2)][k_1 - m_1\omega^2] - k_1^2}$$

Two Degree of Freedom : Forced Vibration



Thank You

Che Ku Eddy Nizwan Bin Che Ku Husin
Faculty of Mechanical Engineering
Universiti Malaysia Pahang

E-mail: eddy@ump.edu.my
Tel: +09-424 6217
Focus Group Website: www.asivr.ump.edu.my