BMM3553 Mechanical Vibrations

Chapter 3: Damped Vibration of Single Degree of Freedom System (Part 1)

by

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Chapter Description

- Expected Outcomes

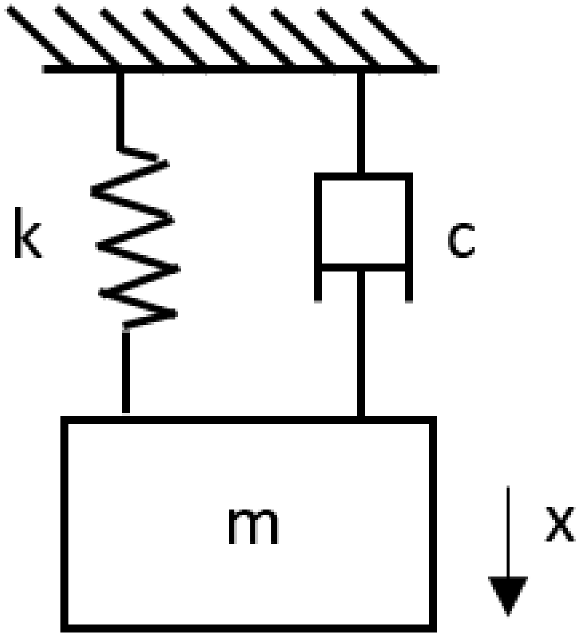
  Students will be able to:
  - Determine the natural frequency for damped free vibration
  - Solve the problem related to damped free vibration

- References

  - Singiresu S. Rao. Mechanical Vibrations. 5th Ed
  - Abdul Ghaffar Abdul Rahman. BMM3553 Mechanical Vibration Note. UMP.
  - Md Mustafizur Rahman. BMM3553 Mechanical Vibration Lecture Note. UMP

SDOF Damped Free Vibration

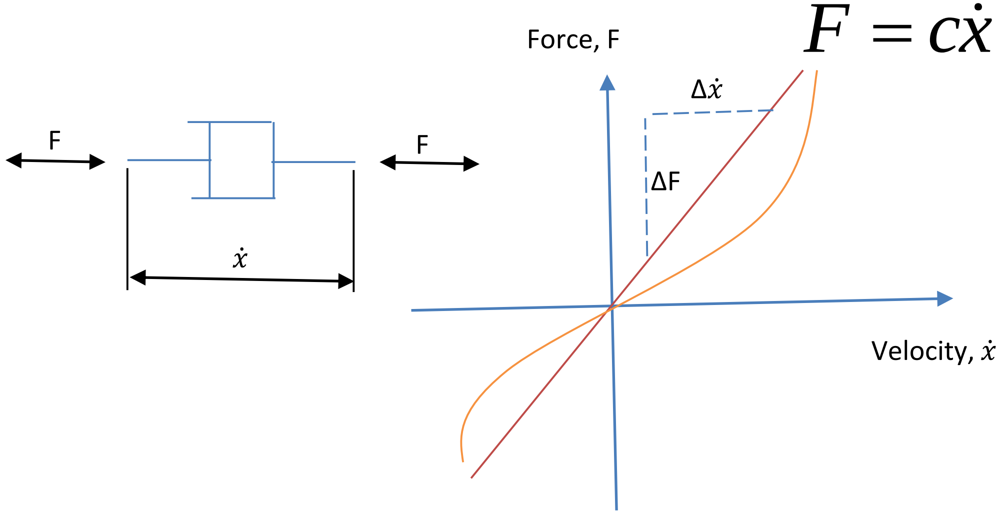
- Given an initial condition, determine the resulting motion.
- Initial condition:
  - x: *Initial position*
  - $\dot{x}$: *Initial velocity*



Viscous Damping Element (Dashpot)

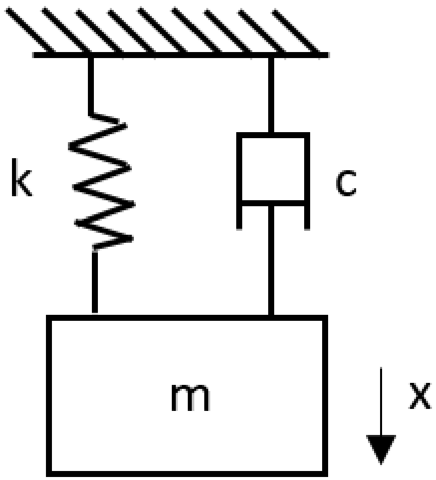
- Damping force is linear and proportional to velocity

$$F = c\dot{x}$$

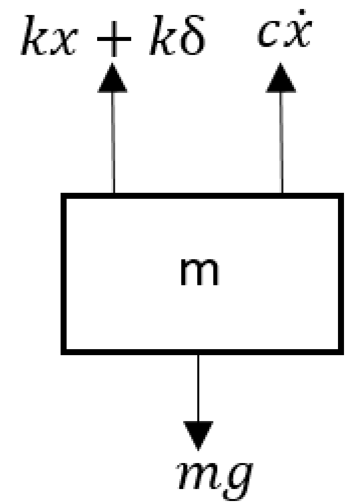


- c is the viscous damping coefficient
- Units: N-sec/m

Maintain Dynamic Equivalent



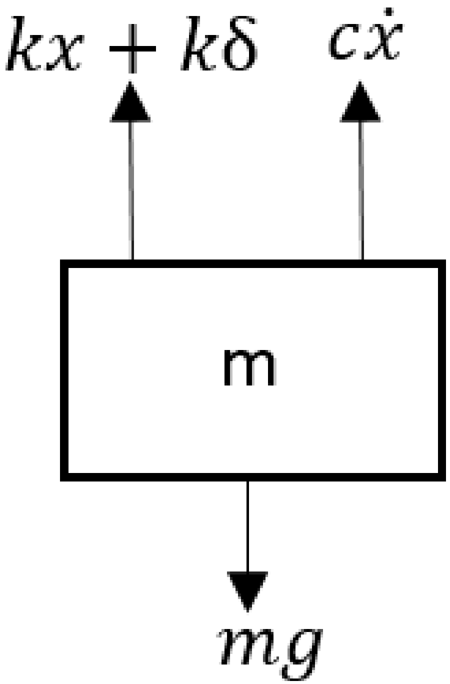
Free Body Diagram



At rest, X = 0 (Static equivalent)

$$mg = k\delta_{st}$$

Maintain Dynamic Equivalent



Apply Newton's 2nd Law

$$\sum F = m\ddot{x}$$

$$\sum F_{x\downarrow +} = m\ddot{x}$$

$$mg - (kx + k\delta) - c\dot{x} = m\ddot{x}$$

Equation of Motion:

$$m\ddot{x} + c\dot{x} + kx = 0$$

Equation of Motion:

$$m\ddot{x} + c\dot{x} + kx = 0$$

- 2nd order differential equation
- Homogeneous
- Linear
- Constant coefficients
- Form of solution:

$$x(t) = A\sin(\omega t + \theta) \quad \text{or} \quad x(t) = Ae^{st}$$

Equation of Motion: $m\ddot{x} + c\dot{x} + kx = 0$

Assume, $x(t) = Ae^{st}$

then $\dot{x}(t) = Ase^{st}$

and $\ddot{x}(t) = As^2e^{st}$

$$mAs^2e^{st} + Asce^{st} + kAe^{st} = 0$$

$$\left(ms^2 + cs + k\right)Ae^{st} = 0$$

for a non-trivial solution

$$ms^2 + cs + k = 0$$

Equation of Motion: $$m\ddot{x} + c\dot{x} + kx = 0$$

$$ms^2 + cs + k = 0$$

$$s_{1,2} = \frac{-c}{2m} \pm \frac{\sqrt{c^2 - 4mk}}{2m}$$

$$x(t) = A_1 e^{s_1 t} + A_2 e^{s_2 t}$$

if s_1 and s_2 are not equal

Thus the general solution is:

$$x(t) = A_1 e^{s_1 t} + A_2 e^{s_2 t}$$

$$= A_1 e^{\left\{ -\frac{c}{2m} + \sqrt{\left(\frac{c}{2m}\right)^2 - \frac{k}{m}} \right\} t} + A_2 e^{\left\{ -\frac{c}{2m} - \sqrt{\left(\frac{c}{2m}\right)^2 - \frac{k}{m}} \right\} t}$$

where A_1 and A_2 are arbitrary constants to be determined from the initial conditions of the system.

Damping Parameters

Critical Damping Constant and Damping Ratio:

$$\left(\frac{c_c}{2m}\right)^2 - \frac{k}{m} = 0$$

$$c_c = 2m\sqrt{\frac{k}{m}} = 2\sqrt{km} = 2m\omega_n$$

The damping ratio, ζ is defined as:

$$\zeta = c / c_c$$

Damped Solution

Define:

$$\omega_n = \sqrt{\frac{k}{m}} = \text{Natural Frequency}$$

$$\zeta = \frac{c}{C_c} = \text{Damping Ratio}$$

$$s_{1,2} = -\zeta\omega_n \pm \sqrt{\left(\zeta^2 - 1\right)}\,\omega_n$$

Thus the general solution is:

$$x(t) = A_1 e^{\left(-\zeta+\sqrt{\zeta^2-1}\right)\omega_n t} + A_2 e^{\left(-\zeta-\sqrt{\zeta^2-1}\right)\omega_n t}$$

Assuming that *ζ ≠ 0*, consider the following 3 cases:

Case1. *Underdamped system*

$$(\zeta < 1 \text{ or } c < c_c \text{ or } c/2m < \sqrt{k/m})$$

For this condition, *(ζ^2-1)* is negative and the roots are:

$$s_1 = \left(-\zeta + i\sqrt{1-\zeta^2}\right)\omega_n$$
$$s_2 = \left(-\zeta - i\sqrt{1-\zeta^2}\right)\omega_n$$

and the solution can be written in different forms:

$$
\begin{aligned}
x(t) &= A_1 e^{\left(-\zeta + i\sqrt{1-\zeta^2}\right)\omega_n t} + A_2 e^{\left(-\zeta - i\sqrt{1-\zeta^2}\right)\omega_n t} \\
&= e^{-\zeta\omega_n t}\left\{ A_1 e^{i\sqrt{1-\zeta^2}\,\omega_n t} + A_2 e^{-i\sqrt{1-\zeta^2}\,\omega_n t} \right\} \\
&= e^{-\zeta\omega_n t}\left\{ C\cos\sqrt{1-\zeta^2}\,\omega_n t + D\sin\sqrt{1-\zeta^2}\,\omega_n t \right\} \\
&= A e^{-\zeta\omega_n t}\sin\left(\sqrt{1-\zeta^2}\,\omega_n t + \phi\right)
\end{aligned}
$$

where *(C,D) and (A,Φ)* are arbitrary constants to be determined from initial conditions.

$$\text{Damped Frequency}, \omega_d = \omega_n\sqrt{1-\zeta^2}$$

$$x(t) = e^{-\zeta\omega_n t}\left\{ C\cos\omega_d t + D\sin\omega_d t \right\}$$

For the initial conditions at *t = 0*

$$C = x_0 \text{ and } D = \frac{\dot{x}_0 + \zeta\omega_n x_0}{\sqrt{1-\zeta^2}\,\omega_n}$$

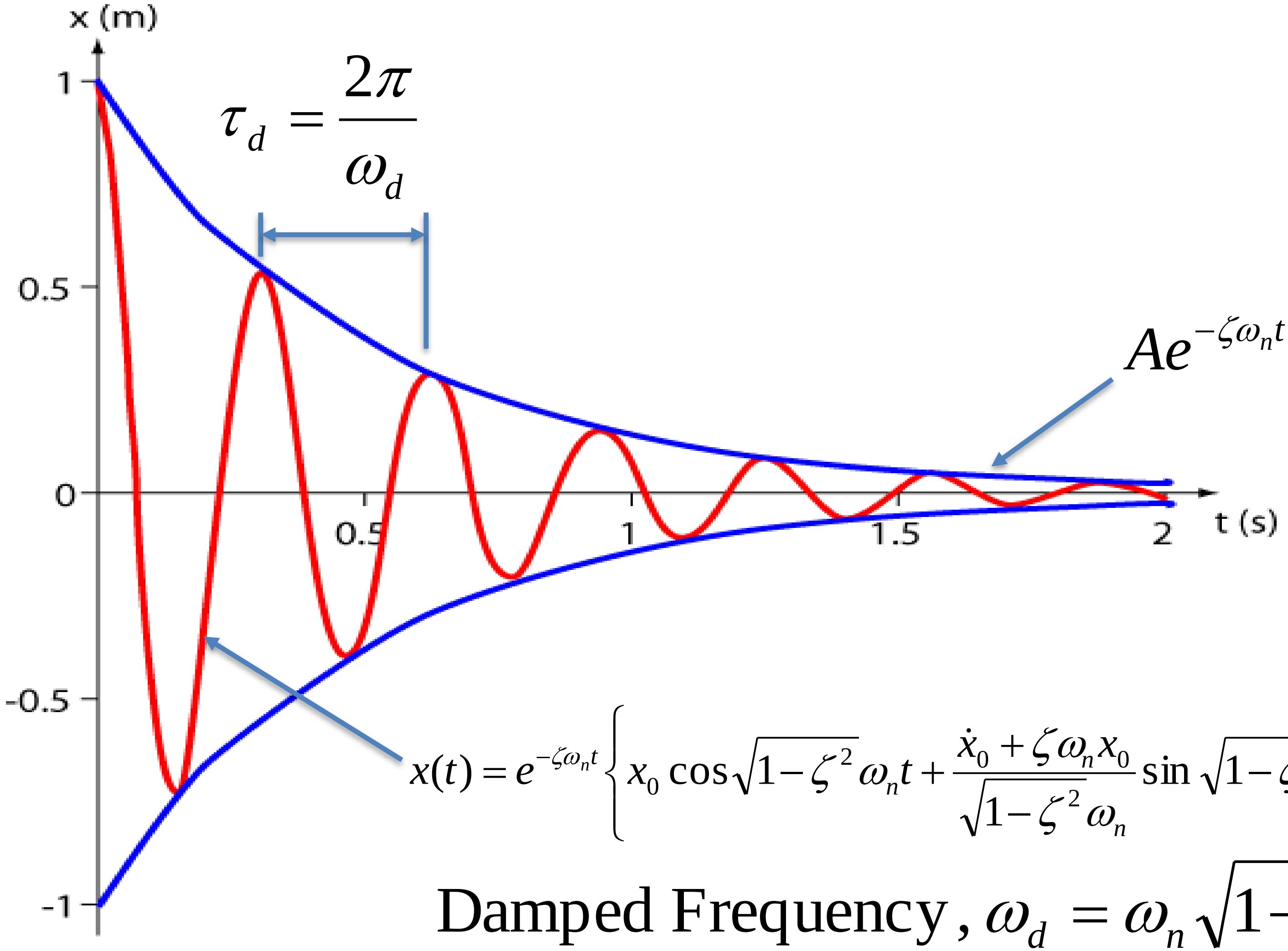
and hence the solution becomes

$$x(t) = e^{-\zeta\omega_n t}\left\{x_0 \cos\sqrt{1-\zeta^2}\,\omega_n t + \frac{\dot{x}_0 + \zeta\omega_n x_0}{\sqrt{1-\zeta^2}\,\omega_n}\sin\sqrt{1-\zeta^2}\,\omega_n t\right\}$$

This equation describes a damped harmonic motion. Its amplitude decreases exponentially with time.

$$\text{Damped Frequency}, \omega_d = \omega_n\sqrt{1-\zeta^2}$$

The *frequency of damped vibration* is:



$$\tau_d = \frac{2\pi}{\omega_d}$$

$$Ae^{-\zeta\omega_n t}$$

$$x(t) = e^{-\zeta\omega_n t}\left\{x_0 \cos\sqrt{1-\zeta^2}\,\omega_n t + \frac{\dot{x}_0 + \zeta\omega_n x_0}{\sqrt{1-\zeta^2}\,\omega_n}\sin\sqrt{1-\zeta^2}\,\omega_n t\right\}$$

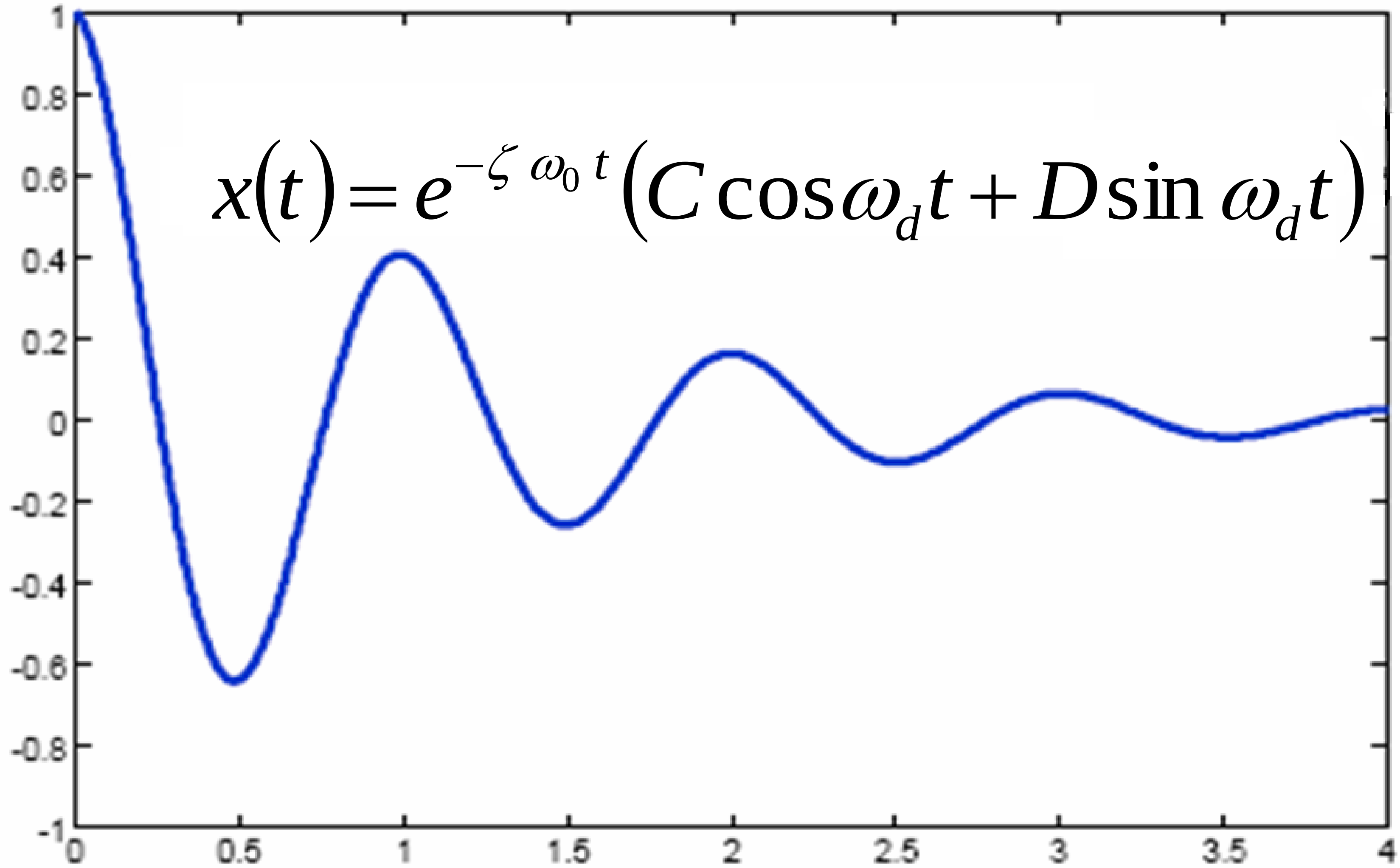
$$\text{Damped Frequency}, \omega_d = \omega_n\sqrt{1-\zeta^2}$$

Image source: https://commons.wikimedia.org/wiki/File:Underdamped_oscillation_xt.png

- **Case 1: ζ<1 Under damped**

(plot of *x*(*t*) vs. time)

$$x(t) = e^{-\zeta\,\omega_0\,t}\left(C\cos\omega_d t + D\sin\omega_d t\right)$$



- **Case 2: ζ=1 Critically damped**

  (Real equal roots)

$$s_{1,2} = -\omega_n$$

$$s_1 = -\omega_n$$

$$s_2 = -\omega_n$$

$$x(t) = A_1 e^{s_1 t} + A_2 t e^{s_2 t} \quad \text{or}$$

$$x(t) = (A_1 + A_2 t) e^{-\omega_n t}$$

- A_1 and A_2 are constants to be found from initial conditions

Case2. *Critically damped system*

$$(\zeta = 1 \text{ or } c = c_c \text{ or } c/2m = \sqrt{k/m})$$

the two roots are:

$$s_1 = s_2 = -\frac{c_c}{2m} = -\omega_n$$

Due to repeated roots,

$$x(t) = (A_1 + A_2 t)e^{-\omega_n t}$$

Application of initial conditions gives:

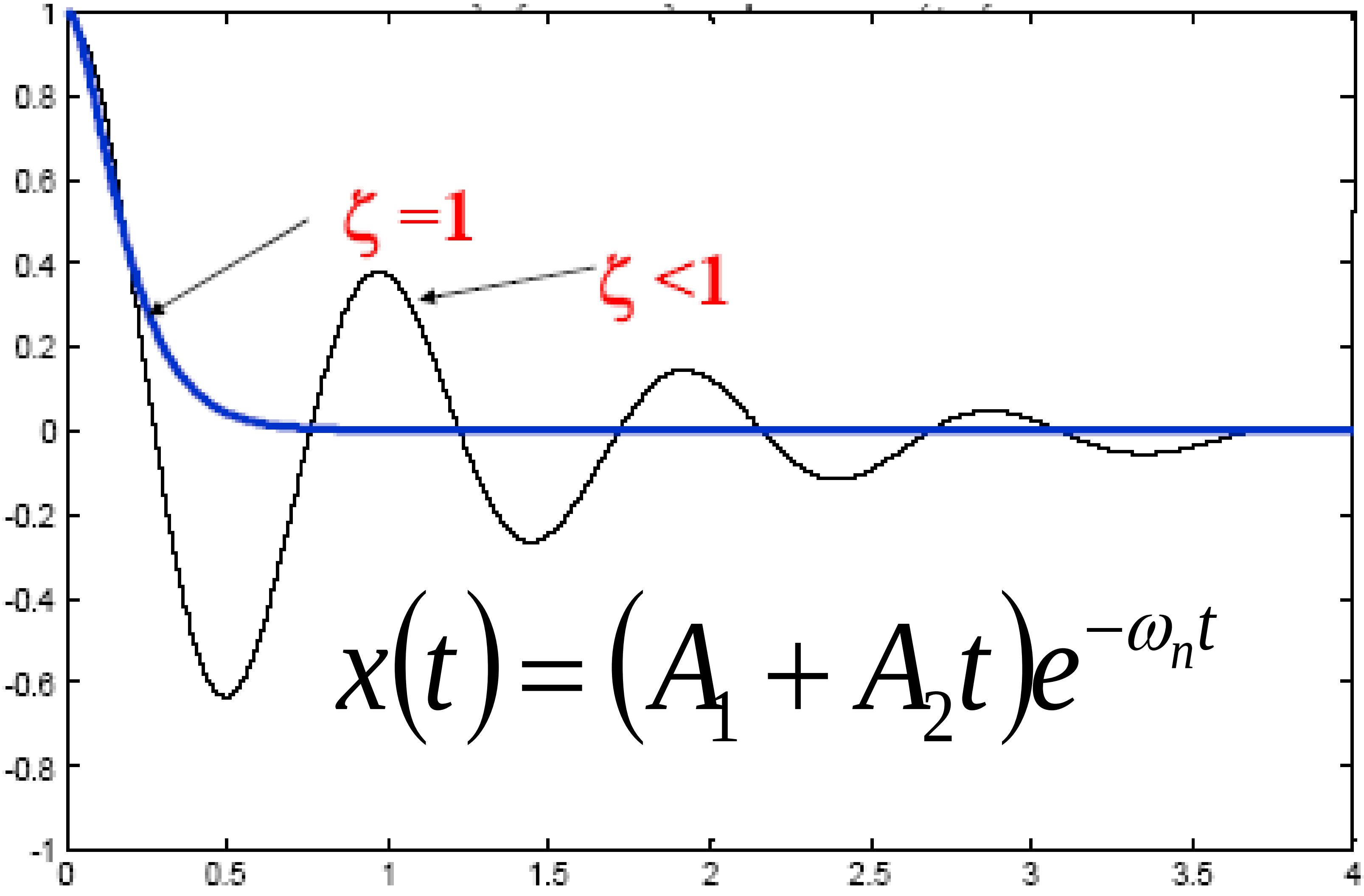
$$A_1 = x_0 \quad \text{and} \quad A_2 = \dot{x}_0 + \omega_n x_0$$

Thus the solution becomes:

$$x(t) = \left[x_0 + \left(\dot{x}_0 + \omega_n x_0\right)t\right]e^{-\omega_n t}$$

- **Case 2: ζ=1 Critically damped**

(Real equal roots)



$$x(t) = (A_1 + A_2 t)e^{-\omega_n t}$$

Case3. *Overdamped system*

$$(\zeta > 1 \text{ or } c > c_c \text{ or } c/2m > \sqrt{k/m})$$

The roots are real

$$s_1 = \left(-\zeta + \sqrt{\zeta^2 - 1}\right)\omega_n < 0$$
$$s_2 = \left(-\zeta - \sqrt{\zeta^2 - 1}\right)\omega_n < 0$$

$$x(t) = A_1 e^{\left(-\zeta + \sqrt{\zeta^2 - 1}\right)\omega_n t} + A_2 e^{\left(-\zeta - \sqrt{\zeta^2 - 1}\right)\omega_n t}$$

For the initial conditions at *t = 0*,

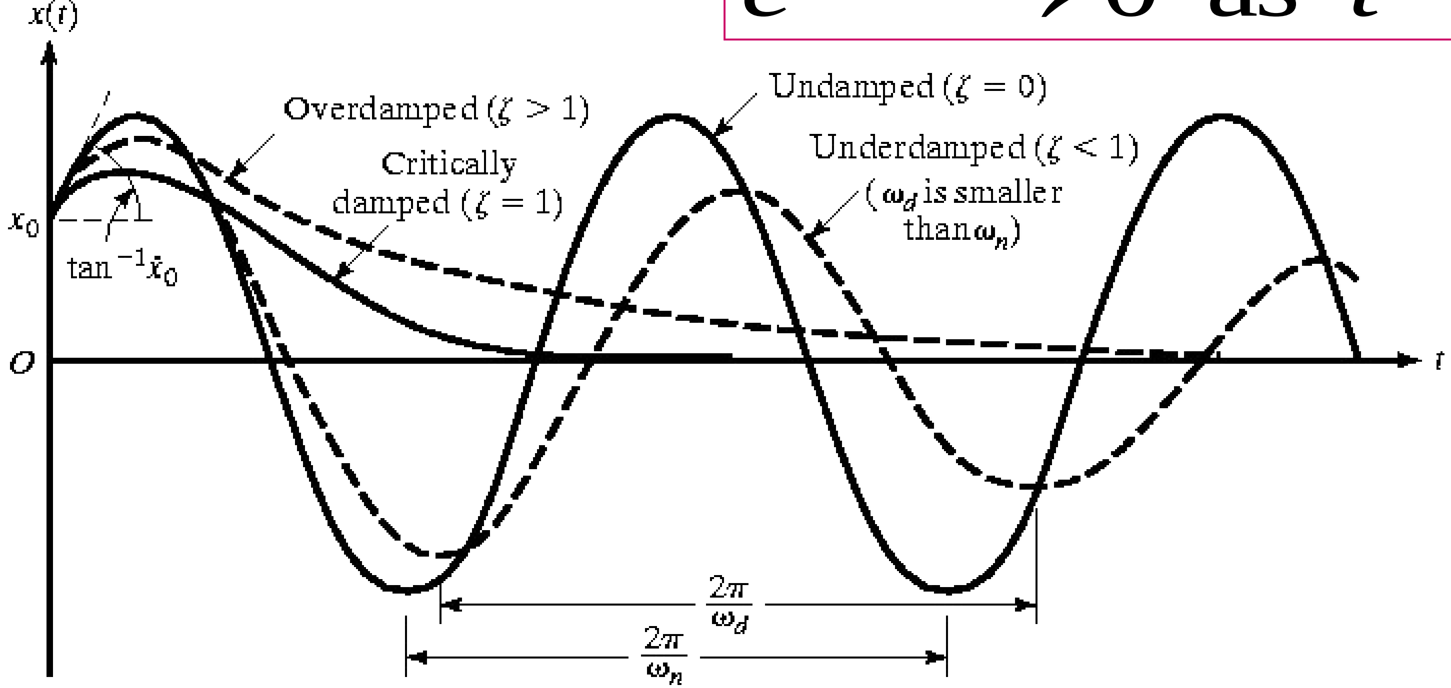
$$A_1 = \frac{x_0\omega_n\left(\zeta + \sqrt{\zeta^2 - 1}\right) + \dot{x}_0}{2\omega_n\sqrt{\zeta^2 - 1}}$$

$$A_2 = \frac{-x_0\omega_n\left(\zeta - \sqrt{\zeta^2 - 1}\right) - \dot{x}_0}{2\omega_n\sqrt{\zeta^2 - 1}}$$

Damped Vibration Response

It can be seen that the motion is aperiodic (i.e., nonperiodic). Since, the motion will eventually diminish to zero.

$$e^{-\omega_n t} \rightarrow 0 \ \text{ as } \ t \rightarrow \infty$$



Comparison of motions with different types of damping

Free Vibration with Viscous Damping

- Logarithmic Decrement:

$$\frac{x_1}{x_2} = \frac{X_0 e^{-\zeta\omega_n t_1}\cos(\omega_d t_1 - \phi_0)}{X_0 e^{-\zeta\omega_n t_2}\cos(\omega_d t_2 - \phi_0)}$$

$$= \frac{e^{-\zeta\omega_n t_1}}{e^{-\zeta\omega_n (t_1+\tau_d)}} = e^{\zeta\omega_n \tau_d}$$

The logarithmic decrement can be obtained

$$\delta = \ln\frac{x_1}{x_2} = \zeta\omega_n\tau_d = \zeta\omega_n\frac{2\pi\zeta}{\sqrt{1-\zeta^2}} = \frac{2\pi}{\omega_d}\cdot\frac{c}{2m}$$

Logarithmic Decrement

Logarithmic decrement : the rate of decrement for free damped vibration amplitude. It is defined as the ratio of any two successive amplitudes .

$$\frac{x_1}{x_{n+1}} = \frac{e^{-\zeta\omega_n t_1}\left(X\sin\left(\omega_d t_1+\phi\right)\right)}{e^{-\zeta\omega_n t_{n+1}}\left(X\sin\left(\omega_d t_{n+1}+\phi\right)\right)}$$

$$\frac{x_1}{x_{n+1}} = \frac{e^{-\zeta\omega_n t_1}}{e^{-\zeta\omega_n (t_1+n\tau_D)}} = e^{n\zeta\omega_n\tau_D}$$

Using: $\tau_D = \frac{2\pi}{\omega_D} = \frac{2\pi}{\omega_n\sqrt{1-\zeta^2}}$

$\frac{x_1}{x_{n+1}}$ becomes

$$\frac{x_1}{x_{n+1}} = e^{\frac{n\zeta\omega_n 2\pi}{\omega_n\sqrt{1-\zeta^2}}} = e^{\frac{n2\pi\zeta}{\sqrt{1-\zeta^2}}}$$

For small damping,

Hence,

$$\delta \approx 2\pi\zeta \qquad \text{if} \qquad \zeta << 1 \qquad (2.86)$$

$$\zeta = \frac{\delta}{\sqrt{(2\pi)^2 + \delta^2}} \qquad (2.87)$$

or

$$\zeta \approx \frac{\delta}{2\pi} \qquad (2.88)$$

Thus,

$$\delta = \frac{1}{m} \ln\left(\frac{x_1}{x_{m+1}}\right) \qquad (2.92)$$

where m is an integer.

Logarithmic Decrement

Given → $\frac{x_1}{x_{n+1}} = e^{\frac{n2\pi\zeta}{\sqrt{1-\zeta^2}}}$ take ln (log$_e$) both sides

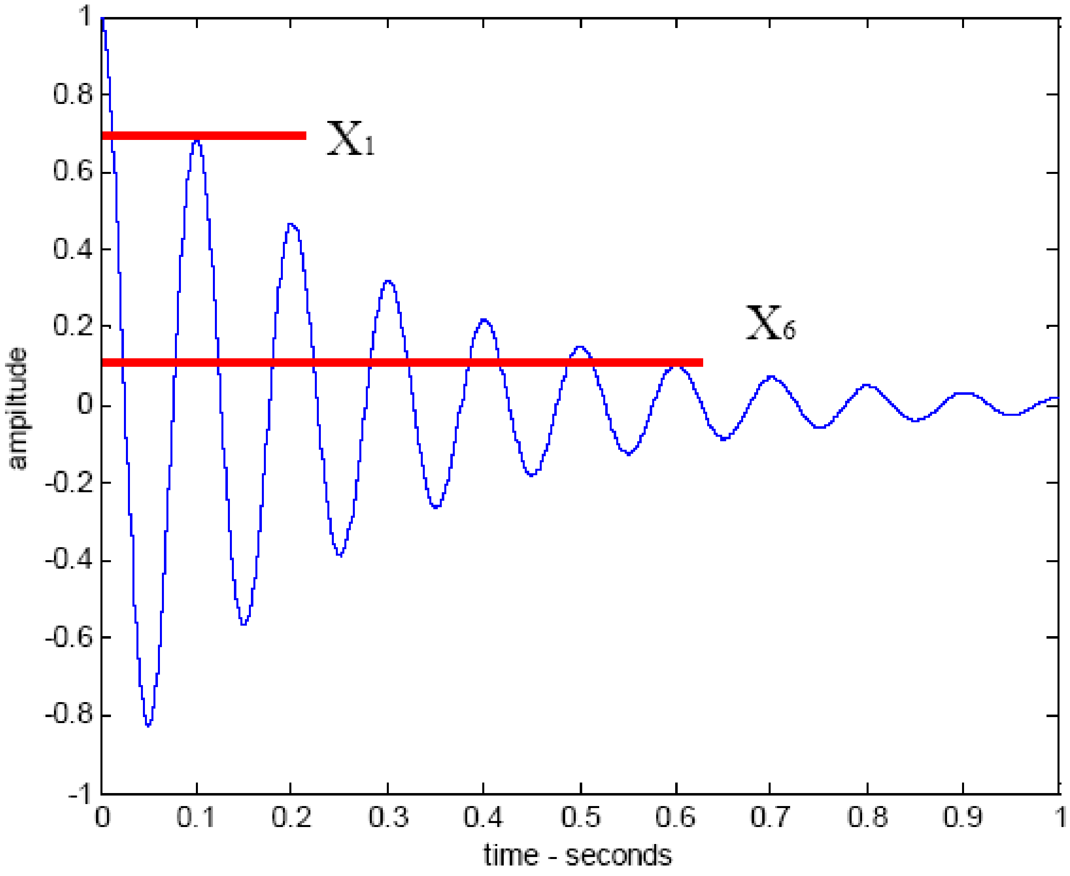
$$\ln\left(\frac{x_1}{x_{n+1}}\right) = \frac{n2\pi\zeta}{\sqrt{1-\zeta^2}} \equiv \text{Log Decrement}$$

$$\frac{1}{n}\ln\left(\frac{x_1}{x_{n+1}}\right) = \frac{2\pi\zeta}{\sqrt{1-\zeta^2}}$$

$$\delta = \frac{1}{n}\ln\left(\frac{x_1}{x_{n+1}}\right) = \frac{2\pi\zeta}{\sqrt{1-\zeta^2}}$$

$$\approx 2\pi\zeta \quad \text{for} \quad \zeta < 0.2$$

Logarithmic Decrement



Example

$$X_1 = 0.68$$

$$X_6 = 0.12$$

$$\frac{1}{5}\ln\left(\frac{0.68}{0.12}\right) \approx 2\pi\zeta$$

$$\zeta = 0.055 \quad 5.5\%$$

Logarithmic Decrement

Damping ratio (for many structural materials) $0.001 \leq \zeta \leq 0.05$

% critical damping $\zeta * 100\%$

$$0.1\% \leq \zeta \leq 5\%$$

Exercise

Problem 2.98 (S.S. Rao 5th Ed)

- The ratio of successive amplitudes of a viscously damped single-degree-of-freedom system is found to be 18:1. Determine the ratio of successive amplitude if the amount of damping is
- (a) double
- (b) halve

solution

$$\ln\frac{x_1}{x_2}=\ln\frac{18}{1}=\frac{2\pi\zeta}{\sqrt{1-\zeta^2}}\Rightarrow\zeta=0.4179$$

(a) If damping is doubled

$$\ln\frac{x_1}{x_2}=\frac{2\pi\zeta_{new}}{\sqrt{1-\zeta_{new}^2}}=\frac{2\pi(0.8358)}{\sqrt{1-(0.8358)_{new}^2}}\Rightarrow\frac{x_1}{x_2}=14265.362$$

(a) If damping is halved

$$\ln\frac{x_1}{x_2}=\frac{2\pi\zeta_{new}}{\sqrt{1-\zeta_{new}^2}}=\frac{2\pi(0.2090)}{\sqrt{1-(0.2090)^2}}\Rightarrow\frac{x_1}{x_2}=3.8296$$

Exercise

Problem 2.103 (S.S. Rao 5th Ed)

- For a spring-mass-damper system, m = 50 kg and k=5000N/m. Find the following:
  - Critical damping constant Cc
  - Damped natural frequency when c = Cc/2
  - Logarithmic decrement.

Solution

$$m = 50\ \text{kg}, \quad k = 5000\ \text{N/m}$$

$$C_c = 2m\omega_n = 2m\sqrt{\frac{k}{m}} = 2\sqrt{km} = 2\sqrt{5000 \times 50} = 1000\ \text{N-s/m}$$

$$c = C_c / 2 = 1000 / 2 = 500\ \text{N-s/m}$$

$$\omega_d = \omega_n\sqrt{1-\zeta^2} = \sqrt{\frac{k}{m}\left(1-\left(\frac{c}{C_c}\right)^2\right)} = \sqrt{\frac{5000}{50}\left(1-\left(\frac{500}{1000}\right)^2\right)} = 8.6603\ \text{rad/s}$$

$$\delta = \frac{2\pi}{\omega_d}\left(\frac{c}{2m}\right) = \frac{2\pi}{8.6603}\left(\frac{500}{2\times 50}\right) = 3.6276$$

REVIEW

For Case $\zeta < 1$

$$x(t) = e^{-\zeta\omega_n t}\left\{x_0 \cos\sqrt{1-\zeta^2}\,\omega_n t + \frac{\dot{x}_0 + \zeta\omega_n x_0}{\sqrt{1-\zeta^2}\,\omega_n}\sin\sqrt{1-\zeta^2}\,\omega_n t\right\}$$

For Case $\zeta = 1$

$$x(t) = \left[x_0 + \left(\dot{x}_0 + \omega_n x_0\right)t\right]e^{-\omega_n t}$$

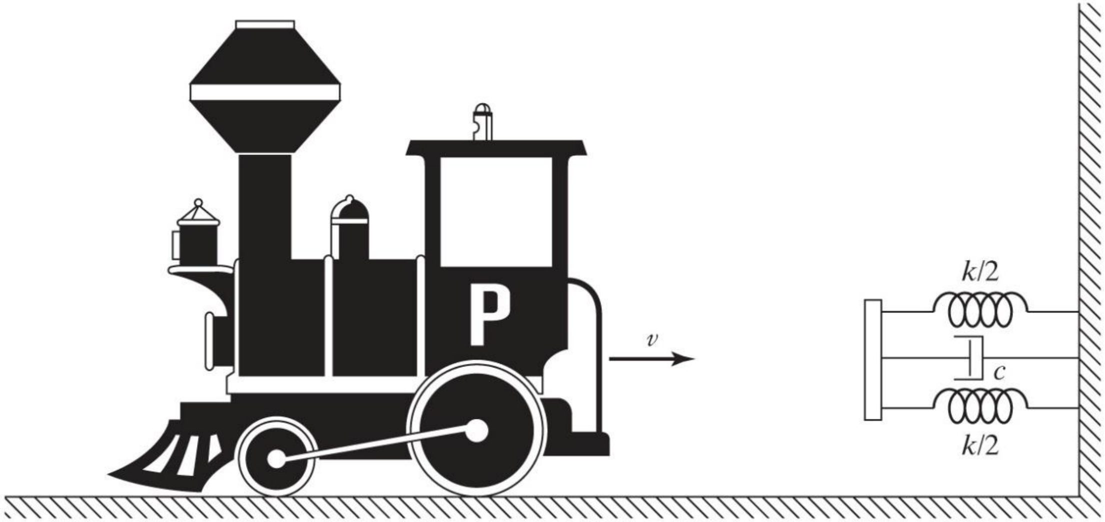
For Case $\zeta > 1$

$$x(t) = \left(\frac{x_0\omega_n\left(\zeta + \sqrt{\zeta^2 - 1}\right) + \dot{x}_0}{2\omega_n\sqrt{\zeta^2 - 1}}\right)e^{\left(-\zeta + \sqrt{\zeta^2 - 1}\right)\omega_n t} + \left(\frac{-x_0\omega_n\left(\zeta - \sqrt{\zeta^2 - 1}\right) - \dot{x}_0}{2\omega_n\sqrt{\zeta^2 - 1}}\right)e^{\left(-\zeta - \sqrt{\zeta^2 - 1}\right)\omega_n t}$$

Exercise

Problem 2.104 (S.S. Rao 5th Ed.)

- A railroad car of mass 2000kg travelling at a velocity v=10m/s is stopped at the end of the tracks by a spring damper system as shown in the figure. If the stiffness of the spring is k=40N/mm and the damping constant is c = 15 N-s/mm, determine (a) the maximum displacement of the car after engaging the spring and damper and (b) the time taken to reach the maximum displacement.



Solution

$$m = 2000\ \text{kg}, \quad v = \dot{x}_0 = 10\ \text{m/s}, \quad k = 40\ \text{N/mm} = 40000\ \text{N/m}$$

$$c = 15\ \text{N-s/mm} = 15000\text{N-s/m}$$

$$\omega_n = \sqrt{\frac{k}{m}} = \sqrt{\frac{40000}{2000}} = 4.4721\ \text{rad/s}$$

$$C_c = 2m\omega_n = 2(2000)(4.4721) = 17884\ \text{N-s/m}$$

$$\zeta = \frac{c}{C_c} = \frac{15000}{17884} = 0.8387$$ (Under damped)

$$\omega_d = \omega_n\sqrt{1-\zeta^2} = 4.4721\sqrt{1-(0.8387)^2} = 2.4346\ \text{rad/s}$$

$$\tau_\text{d} = \frac{2\pi}{\omega_d} = \frac{2\pi}{2.4346} = 2.5807\ \text{sec}$$

For $x_0 = 0, \quad \text{and } \dot{x}_0 = 10 \text{ m/s}$

$$x(t) = e^{-\zeta\omega_n t}\left\{x_0 \cos\sqrt{1-\zeta^2}\,\omega_n t + \frac{\dot{x}_0 + \zeta\omega_n x_0}{\sqrt{1-\zeta^2}\,\omega_n}\sin\sqrt{1-\zeta^2}\,\omega_n t\right\}$$

$$x(t) = e^{-\zeta\omega_n t}\left\{\frac{\dot{x}_0}{\sqrt{1-\zeta^2}\,\omega_n}\sin\sqrt{1-\zeta^2}\,\omega_n t\right\}$$

At $x_{\max}$, $\omega_n t = \frac{\pi}{2}$ and $\sin\omega_n\sqrt{1-\zeta^2}\,t = 1$

$$x_{\max} = e^{-(0.8387)\left(\frac{\pi}{2}\right)}\left\{\frac{10}{\sqrt{1-(0.8387)^2}\,(4.4721)}(1)\right\} = 1.1001\text{ m}$$

$$\omega_n t = \frac{\pi}{2} \Rightarrow t = \frac{\pi}{2\omega_n} = \frac{\pi}{2\times 4.471} = 0.3513\text{ sec}$$

Thank You

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