DIGITAL SIGNAL PROCESSING

Chapter 9
Discrete Fourier Transform (DFT)

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Discrete Fourier Transform

- Aims
  - To explain the characteristic of the discrete-time signal in frequency domain and perform analysis in frequency domain.

- Expected Outcomes
  - Upon completion of the topic, students should be able to convert and analyze the discrete-time signals in frequency-time domain and obtain the response of the discrete-time signal using Discrete Fourier Transform (DFT) technique.

Definition of DFT

- Discrete-time signal, x(n), can be analyzed in frequency domain by using analysis method called **Discrete Fourier Transform or DFT**.
- The continuous Fourier Transform is defined as below:

$$F(\omega) = \int_{-\infty}^{\infty} f(t)\exp(-j\omega t)dt$$

$$f(t) = (1/2\pi) \int_{-\infty}^{\infty} F(\omega)\exp(j\omega t)dt$$

- However, this integral equation of Fourier Transform cannot implement frequency analysis since cannot handle nature of continuous manually which just able be done by Computer, and beside that,
  - Continuous nature can be handled by Computer
  - Require continuous limit for integration where merely finite length sequences that can be processed by computer.

DFT Properties

- Let x(n) be a finite length sequences.

  Thus, the N-point DFT of x(n) defined as X(k) is :

$$\mathbf{X(k)} = \sum_{n=0}^{N-1} x(n)\, e^{-j2\pi nk/N}, \quad k = 0,1, \ldots, N-1$$

- k represents the transformation components
- n is counter of the finite length sequence

- Thus X(k) will become a complex and has real & imaginary components, hence, the components of X(k) is defined as:

  X(k) = real(k) + imaginary, j(k)

DFT Properties

- The are 4 properties of DFT:

1. **Periodicity**

   If DFT signal, X(k) is the N-point DFT of x(k), thus,

   $x(n+N) = x(n)$, for all n
   $X(k+N) = X(k)$, for all k

2. **Linearity**

   If the combination of DFT signal, $X_1(k)$ and $X_2(k)$ of the DTS signal, $x_1(n)$ and $x_2(n)$, thus,

   $$a x_1(n) + b x_2(n) \xleftrightarrow{\text{DFT}} aX_1(k) + bX_2(k)$$

DFT Properties

3. **Circular Shifting**

   The sequence of the DTS signal, $x(n)$ of length N and the DFT of the sequence is X(k) with length of N, then, the sequence, $x'(n)$ obtained from $x(n)$ by shifting $x(n)$ cyclically by m units. Then,

$$x'(n) \overset{DFT}{\longleftrightarrow} X(k)e^{-j2\pi km/N}$$

4. **Parseval's Theorem**

$$\text{if } x(n) \overset{DFT}{\longleftrightarrow} X(k) \text{ and}$$

$$y(n) \overset{DFT}{\longleftrightarrow} Y(k)$$

$$\text{thus, } \sum_{n=0}^{N-1} x(n)y^*(n) = 1/N \sum_{k=0}^{N-1} X(k)Y^*(k)$$

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DFT & z-Transform

- The z-transform of the sequence, $x(n)$ is given by:

$$X(z) = \sum_{n=-\infty}^{\infty} \boldsymbol{x}(\mathbf{n})\, z^{-n} \quad \text{, ROC include unit circle}$$

by defining $z^k = e^{j2\pi k/N}$, k = 0, 1, 2, ..., N-1

$$X(k) = X(z)\big|_{z^k = e^{j2\pi k/N}} \quad , k = 0,1,2, \ldots, N-1$$

$$= \sum_{n=-\infty}^{\infty} \boldsymbol{x}(\mathbf{n})\, e^{-j2\pi nk/N}$$

where $\omega_k = 2\pi k/N$, k = 0,1,2,...,N-1

DFT Example

- **EXAMPLE 1:**
  Find the DFT for the following finite length sequence,
  $x(n) = \{ ¼, ¼, ¼ \}$
  Solution :

1. Determine the sequence length, N
   N = 3, k = 0,1,2
2. Use DFT formula to determine X(k)

$$X(k) = \sum_{n=0}^{N-1} x(n)\, e^{-j2\pi nk/N}, \quad k = 0, 1, 2$$

X(0) = ¼ + ¼ + ¼ = **¾ (0.75)**

$X(1) = ¼ + ¼e^{-j2\pi/3} + ¼e^{-j4\pi/3}$

$= ¼ + ¼ [\cos(2\pi/3) - j\sin(2\pi/3)] + ¼ [\cos(4\pi/3) - j\sin(4\pi/3)]$

= ¼ + ¼ [-0.5 – j0.866] + ¼ [-0.5 + j0.866]

= 0.25-0.25 = **0**

DFT Example

- Continued from **Example 1:**

$X(2) = ¼ + ¼\, e^{-j4\pi/3} + ¼\, e^{-j8\pi/3}$

$= ¼ + ¼\,[\cos(4\pi/3) – j\sin(4\pi/3)] +$

$¼\,[\cos(8\pi/3) – j\sin(8\pi/3)]$

$= 0.25 + 0.25\,[-0.5 + j0.866] + 0.25[-0.5 –$

$j0.866]$

$= \mathbf{0}$

Thus,

X(k) = { 0.75, 0, 0}

DFT Example

- **Examples 2:**

  Given the following the finite length sequences,

  $x(n) = \{1,1,2,2,3,3\}$

  Perform frequency analysis of the signal using DFT technique for the sequences.

  Solution:

1. Determine the sequence length, N = 6.
2. Use DFT formula to determine X(k).

$$X(k) = \sum_{n=0}^{N-1} \boldsymbol{x(n)}\, e^{-j2\pi nk/N}, \quad k = 0,1,2,3,4,5$$

X(0) = 12, X(1) = -1.5 + j2.598

X(2) = -1.5 + j0.866, X(3) = 0

X(4) = -1.5 – j0.866, X(5) = -1.5 – j2.598

Thus, the DFT sequences are;

X(k) = {12, -1.5 + j2.598, -1.5 + j0.866, 0, -1.5 – j0.866, -1.5 – j2.598}

Convolution in DFT

- To perform convolution for DFT, need to do the following:
  1. Find N-point DFT of the sequence h(n) and x(n).
  2. Multiply DFT to form Y(k) = H(k)X(k)
  3. Perform inverse DFT to obtain y(n).

Convolution in DFT : Example

- **Examples 3:**

  Find the DFT for the convolution of 2 sequences of the DTS signals as describe below:

  $x_1(n)$ = {2, 1, 2, 1}, $x_2(n)$ = {1, 2, 3, 4}

  Solution:

1. Determine the sequence length for each sequence. Here, the length of both sequences, N = 4. Thus, the counter for DFT sequence, k = 0,1,2,3.
2. Next, perform DFT for each sequence.

   (i) $X_1(0) = \sum_{n=0}^{3} \boldsymbol{x}\mathbf{(n)}\, e^{-j2\pi nk/3} = 6$, repeat for $X_1(1) = 0$, $X_1(2) = 2$, $X_2(3) = 0$

   $X_1(k) = \mathbf{\{6,0,2,0\}}$

   (ii) $X_2(0) = \sum_{n=0}^{3} \boldsymbol{x}\mathbf{(n)}\, e^{-j2\pi nk/3}$ 10, repeat for $X_2(1) = -2+j2$, $X_2(2) = -2$, $X_2(3) = -2-j2$

   $X_2(k) = \mathbf{\{10,-2+j2,-2,-2-j2\}}$

3. Perform Convolution either using graphical or mathematical method, to obtain:

   $X_3(k) = X_1(k)\, X_2(k) = \mathbf{\{60, 0, -4, 0\}}$

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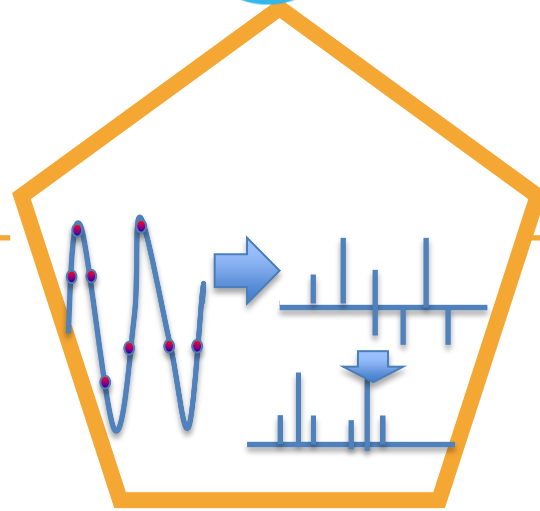

1 To obtain discrete-time signals

2 To convert the discrete-time signals into frequency domain using DFT

3 To obtain the properties of the DFT & perform convolution

4 To calculate the DFT of the discrete-time signal

5 To analyze the discrete-time signal response and characteristic in frequency domain by using DFT method

Conclusion of The Chapter

- Able to obtain the response of the discrete-time signal using DFT technique.
- Able to perform frequency analysis of the discrete-time signal using DFT technique.
- Able to perform discrete-signal convolution using DFT technique.

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