

# DIGITAL SIGNAL PROCESSING

## Chapter 6 IIR Filter Design



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# Filter design

- Aims
  - To explain type of IIR filter, filter design steps, and filter conversion (analogue to digital and vice versa)
- Expected Outcomes
  - The outcome of the topic should have student to be able to design IIR digital filter based on the filter specifications, and to convert analogue filter to digital filter and vice versa using Bilinear transformation method and Impulse Invariant method.



# Definition of Filter

- ❑ Filter is required in the digital signal processing to filter the raw input signals to the desired frequency and suppress noise in signal processing.
- ❑ Filter consists of Finite Impulse Response (FIR) and Infinite Impulse Response Filter (IIR).
- ❑ There are four type filter such as Low-pass, High-pass, Band-pass and Band-stop filter.



# IIR Filter

- ❑ It is a system where the output of the system not only depend on the input signals but the past values of the output signals.
- ❑ The system only has both zeros and poles.
- ❑ The system has feedback.
- ❑ The stability of the system depends on its poles.
- ❑ Example of the difference equation that can describe the system;

$$\rightarrow y[n] = \frac{1}{2} y[n-1] + 2x[n] + x[n-1]$$



# IIR Filter Specifications

- ❑ The process of filter design begins with filter specifications which include the filter characteristics (Low-pass, high-pass, band-pass, band-stop filter), filter type, passband frequency, stopband frequency, transition width frequency, sampling frequency and filter length.
- ❑ The second step is obtain filter response,  $H(\omega)$ .
- ❑ Third step is to find the filter coefficient and acceptable filter.
- ❑ The last step is to implement filter coefficient and choose appropriate filter structure for filter implementation.



# Mapping of s-plane & z-plane

- The relationship between frequency domain and time domain by **backward difference derivative**, is shown below:

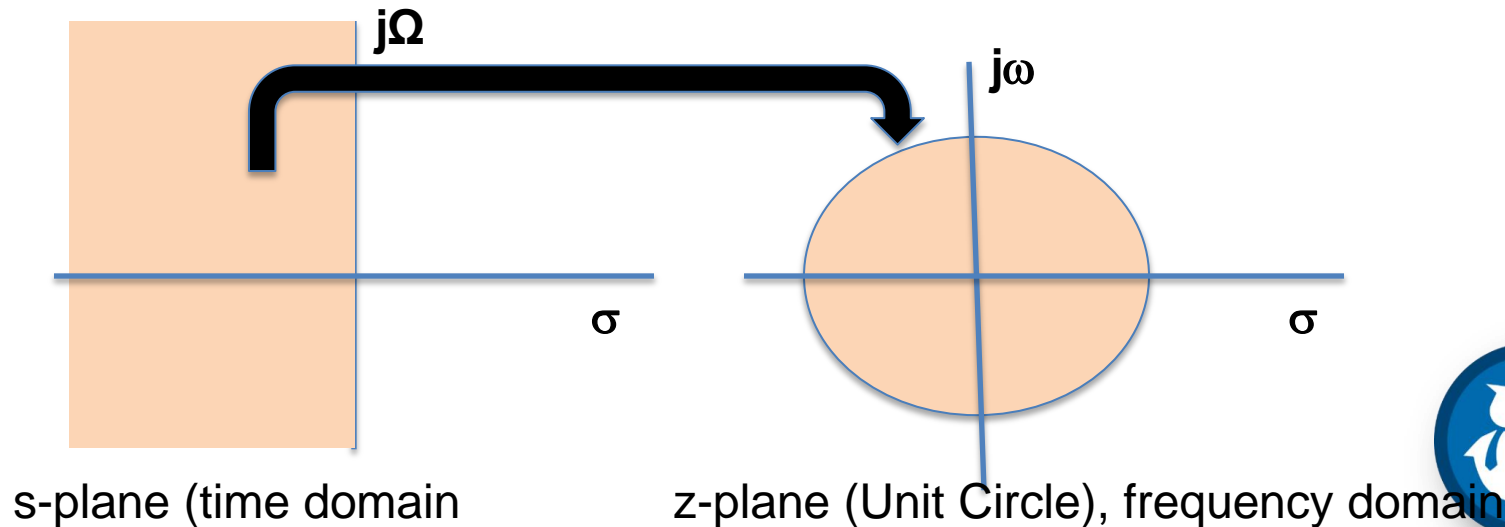
$$s = (1 - z^{-1}) / T_s$$

- The mapping of s-plane (analog) to z-plane (digital) is done by:

$$z = 1 / (1 - sT) \text{ or } 1 / (1 - j\Omega T_s), H(s) = s, \Rightarrow H(z) = (1 - z^{-1}) / T_s$$

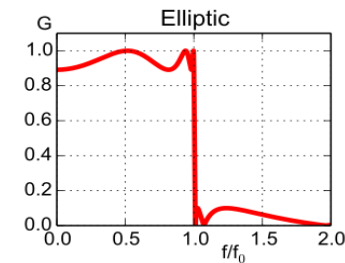
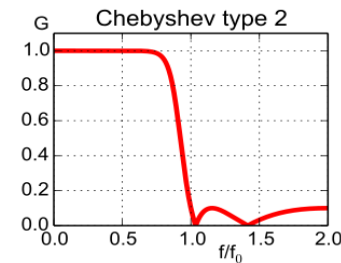
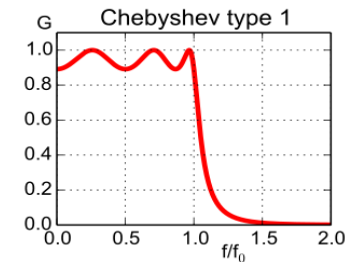
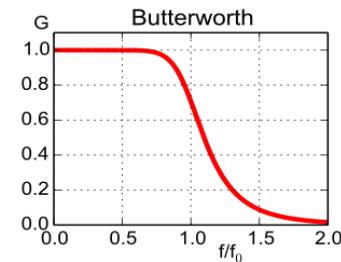
- The mapping of the s-plane to the z-plane is defined as  $z = re^{sT}$ .

Since  $s = \sigma + j\Omega$ ,  $z = e^{\sigma T} e^{j\Omega T}$ , Thus,  $r = e^{\sigma T}$ ,  $z = e^{j\Omega T}$ ,  $\omega = \Omega T$  (Unit Circle) as shown in diagram below;



# IIR Filter Type & Characteristics

- There are 4 common IIR filter designs:
  1. Butterworth
    - (As the Filter Order,  $N$  increases, the transition band becomes narrower).
  - 2a. Chebyshev Type I (ripple at pass-band)
  - 2b. Chebyshev Type II (ripple at stop-band)
  3. Elliptic (ripple at both pass-band & stop-band)
  4. Bessel (linear-phase response over the pass-band and exhibit larger transition bandwidth)
- The analog filter will be mapped to digital filter using transformation of s-domain to z-domain.
- 2 methods to convert the analog filter to digital filter and vice versa:
  1. Impulse Invariance method
  2. Bilinear Transformation method



(Source: [https://upload.wikimedia.org/wikipedia/commons/thumb/0/0f/Filters\\_order6](https://upload.wikimedia.org/wikipedia/commons/thumb/0/0f/Filters_order6))



# IIR Filter Specifications: Butterworth filter

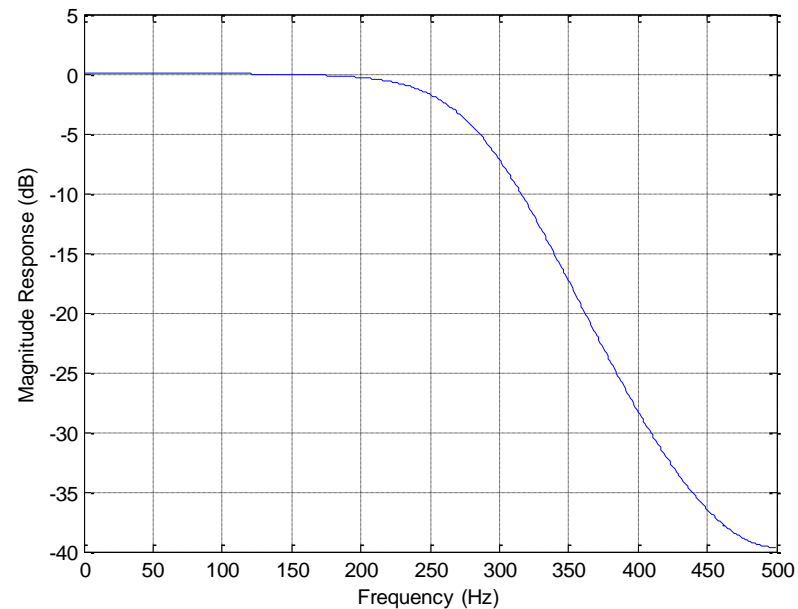
- ❑ The example of **IIR Butterworth Low-Pass Filter** at sampling frequency of 1 kHz and cut-off frequency of 300 Hz and filter order of 5 using Impulse Invariance method is illustrated by the diagram below:

## Amplitude Parameters;

- ❑ Passband ripple =  $\delta_p$
- ❑ Stopband ripple =  $\delta_s$
- ❑ Discrimination Factor,  
$$d = \sqrt{\frac{(1-\delta_p)^{-2} - 1}{(\delta_s^{-2} - 1)}}$$
 or  $\epsilon / \sqrt{A^2 - 1}$

## Frequency Parameters;

- ❑ Sampling frequency =  $F_s$
- ❑ Selectivity Factor,  $k = \Omega_p / \Omega_s$
- ❑ The Filter Order, N is defined as:  
$$N \geq \log d / \log k$$





# Butterworth IIR Low-pass filter : Example

Design the Butterworth Low-pass Filter to meet the following specifications:

$$f_p = 6 \text{ kHz}, f_s = 10 \text{ kHz}, \delta_p = \delta_s = 0.1$$

## Solution :

1. Calculate  $d$  and  $k$ ,

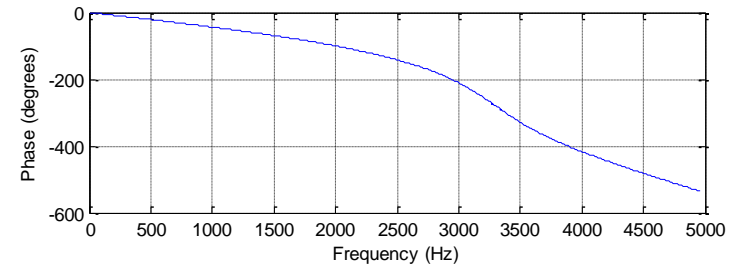
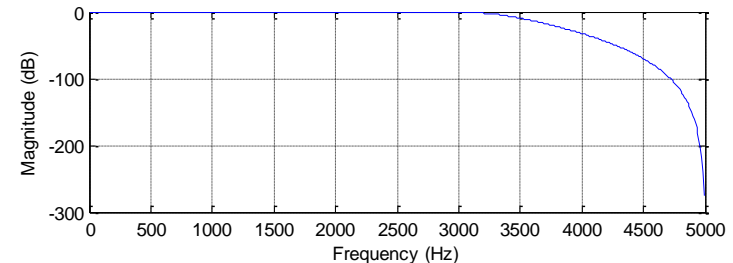
$$d = \sqrt{\frac{((1-\delta_p)^{-2} - 1)}{(\delta_s^{-2} - 1)}} = \mathbf{0.0487}$$

$$k = \Omega_p / \Omega_s = \mathbf{0.6}$$

2. Calculate Filter Order,  $N$

$$N \geq \frac{\log d / \log k}{\log 0.6} = \log (0.0487) / \log (0.6) = 5.9$$

Thus,  $\mathbf{N = 6}$



# Butterworth IIR Low-pass filter : Example

3. To determine the cut-off frequency,  $\Omega_c$

$$f_p[(1-\delta_p)^{-2} - 1]^{-1/2N} = 6770$$

$$f_s[\delta_s^{-2} - 1]^{-1/2N} = 6819$$

$$6770 \leq \Omega_c \leq 6819$$

4. The pole position are:

$$s_k = \Omega_c e^{j[\pi/2 + (2k+1)\pi/12]}, k = 0, 1, \dots, 5$$

5. The Transfer Function is :

$$\begin{aligned} H_a(s) = & 1 / [s^6 + 3.8637s^5 + 7.4641s^4 \\ & + 9.1416s^3 + 7.4641s^2 + \\ & 3.8637s + 1 \end{aligned}$$



# Butterworth IIR Band-pass filter : Example

Design the **Butterworth IIR Band-pass Filter** to meet the following Filter specifications by using **Bilinear Transformation method**.

$$f_{p1} = 200 \text{ Hz}, f_{p2} = 300 \text{ Hz}, f_{s1} = 50 \text{ Hz}, \\ f_{s2} = 450 \text{ Hz}, A_p = 3 \text{ dB}, A_s = 20 \text{ dB}, F_s = 1000 \text{ Hz}$$

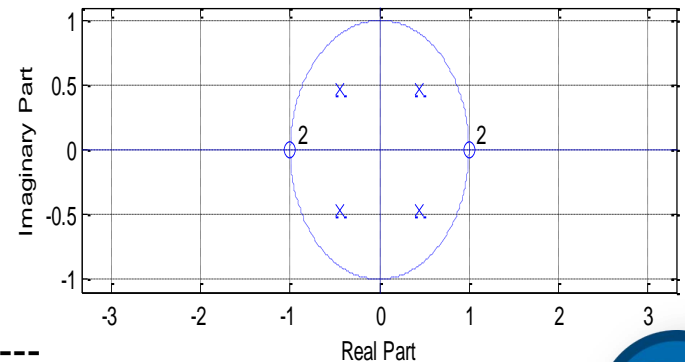
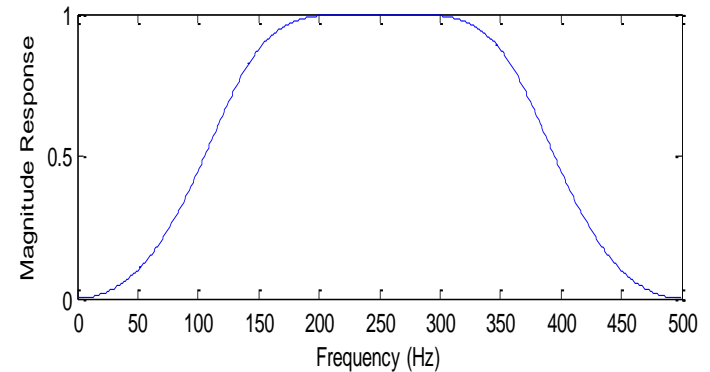
**Solution :**

1. Calculate  $\delta_p$  and  $\delta_s$   
 $\delta_p = 1 - \text{antilog}(-3/20) = 0.2921$   
 $\delta_s = \text{antilog}(-20/20) = 0.1$
2. Calculate  $d$  and  $k$ ,  
 $d = \text{sqrt}[\frac{(1-\delta_p)^2 - 1}{(\delta_s^{-2} - 1)}] = \mathbf{0.1}$   
 $k = \Omega_{p2} / \Omega_{s2} = 300 / 450 = \mathbf{0.67}$
3. Calculate Filter Order,  $N$   
 $N \geq \log d / \log k = \log(0.1) / \log(0.67)$   
 $\mathbf{N = 6}$
4. **The transfer function,  $H(s)$**

$$0.2857 s^4 - 0.5714 s^2 + 0.2857$$

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$$s^4 + 8.882e-16 s^3 + 0.02882 s^2 - 8.327e-17 s + 0.1717$$



# IIR Filter Specifications: Chebyshev filter

- There are 2 types of Chebyshev Filters; Type 1 and Type 2.
- Type 1 Chebyshev Filter is all-pole with an equiripple passband and a monotonically decreasing stopband.
- The magnitude of Frequency Response of type 1 Chebyshev Filter is:

$$|H_a(j\Omega)|^2 = 1 / [1 + \varepsilon^2 T_N^2(\Omega/\Omega_p)]$$

$N$  = Filter Order

$\Omega_p$  = passband cut-off frequency

$\varepsilon$  = parameter that controls passband ripple amplitude



# IIR Filter Specifications: Chebyshev filter

- The discrimination and selectivity factor formula is same as Butterworth IIR filter.
- The Filter Order, N is determined by :

$$N \geq [\cosh^{-1} (1/d) / \cosh^{-1} (1/k)]$$

- The Transfer Function is determined by:

$$G_a(s) = H_a(s)H_a(-s) = 1 / [1 + \varepsilon^2 T_N(s/j\Omega_p)]$$

$$\varepsilon = [(1-\delta_p)^{-2} - 1]^{1/2}$$



# IIR Filter Specifications: Chebyshev filter

- The parameter,  $\varepsilon$  which control the pass-band ripple can be defined as

$$\varepsilon^2 = 1 / [(1 - \delta_p)^2] - 1$$

- The parameter,  $\beta$  which is related to  $\varepsilon$  is defined as:

$$\beta = [(\text{sqrt}(1 + \varepsilon^2) + 1) / \varepsilon]^{1/N}, \text{ N is filter length/order}$$

- $r_1 = \Omega_p (\beta^2 + 1) / 2\beta$  and  $r_2 = \Omega_p (\beta^2 - 1) / 2\beta$

- The angular position that can be determined by:

$$\theta_k = \pi/2 + (2k + 1)\pi/2N, \quad k = 0, 1, 2, \dots, N-1$$

- The coordinate of the poles will be:

$$x_k = r_2 \cos \theta_k, \quad y_k = r_1 \sin \theta_k, \quad k = 0, 1, 2, \dots, N-1$$

Thus, the poles are:  $\mathbf{x}_k + \mathbf{j}y_k$



# Chebyshev IIR low-pass filter : Example

Design a Chebyshev Type 1 low-pass filter to meet the following spec:

$$f_p = 6 \text{ kHz}, f_s = 10 \text{ kHz}, \delta_p = \delta_s = 0.1$$

## Solution :

1. Calculate  $d$  &  $k$ ,

$$d = \sqrt{((1-\delta_p)^{-2} - 1) / (\delta_s^{-2} - 1)} = \mathbf{0.0487}$$

$$k = \Omega_p / \Omega_s = \mathbf{0.6}$$

2. Calculate the Filter Order,  $N$

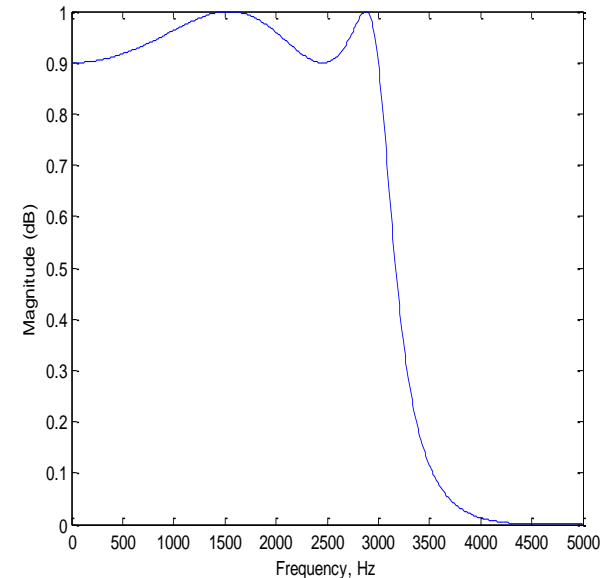
$$N \geq \cosh^{-1}(1/d) / \cosh^{-1}(1/k) \geq 3.38 = \mathbf{4}$$

3. Calculate pass-band ripple controller,  $\varepsilon$

$$\varepsilon = [(1-\delta_p)^{-2} - 1]^{1/2} = \mathbf{0.4843}$$

4. The magnitude of the frequency response of the Chebyshev Type I low-pass filter:

$$\begin{aligned} |H_a(\Omega)|^2 &= 1 / [1 + \varepsilon^2 T_N^2(\Omega/\Omega_p)] = \\ &= 1 / [1 + (0.4843)^2 \Omega / (12000\pi)] = \\ &= \mathbf{1 / [1 + 6.22 \times 10^{-6} \Omega]} \end{aligned}$$



# Elliptic IIR band-pass filter : Example

Design a Elliptic band-pass filter to meet the following spec:

Pass-band frequency,  $f_p = 20.5 - 23.5$  kHz

Stop-band frequency,  $f_s = 0 - 19$  kHz,  $25 - 50$  kHz

Pass-band ripple,  $\delta_p \leq 0.25$  dB

Stop-band attenuation  $> 45$  dB

Sampling frequency,  $F_s = 100$  kHz

## Solution :

1. Calculate  $d$  &  $k$ ,

$$d = \sqrt{\frac{((1-\delta_p)^{-2} - 1)}{(\delta_s^{-2} - 1)}} = \mathbf{0.0284}$$

$$k = \Omega_p / \Omega_s = 20.5/25 = \mathbf{0.82}$$

$$q = 0.5 (1 - (1-k^2)^{1/4}) / (1 + (1-k^2)^{1/4}) = \mathbf{0.0693}$$

2. Calculate the Filter Order,  $N$

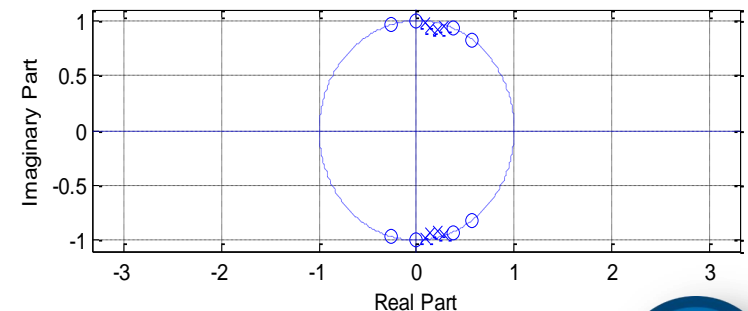
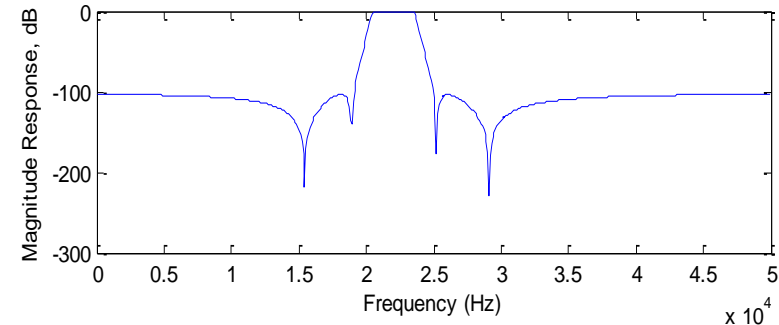
$$N \geq \log(16/d^2) / \log(1/q) \geq 3.7 = \mathbf{4}$$

3. System transfer function is;

$H(s) =$

$$0.006128 s^8 - 0.008248 s^7 + 0.02362 s^6 - 0.02211 s^5 + 0.03505 s^4 - 0.02211 s^3 + 0.02362 s^2 - 0.008248 s + 0.006128$$

$$s^8 - 1.448 s^7 + 4.483 s^6 - 4.221 s^5 + 6.647 s^4 - 3.946 s^3 + 3.919 s^2 - 1.183 s + 0.7634$$





# Filter Conversion

- The design of the digital IIR Filter,  $H(z)$  from analog filter,  $H_a(s)$  require a mapping of s-plane to z-plane.
- The point in the left half s-plane should map to points inside the unit circle to preserve the stability of the analog filter.
- One method to map the left half s-plane to inside unit circle of z-plane is called :

***BILINEAR TRANSFORMATION***



# Filter Conversion

- The Bilinear Transformation is defined by:

$$s = 2/T_s [(1 - z^{-1})/(1 + z^{-1})]$$

- Thus, the Transfer Function of Digital Filter is defined as :

$$H(z) = H_a 2/T_s [(1 - z^{-1})/(1 + z^{-1})]$$

- The relation s-plane & z-plane is:

$$s = \sigma + j\Omega \quad \& \quad z = re^{j\omega}$$



# Filter Conversion

- The mapping process is highly non-linear. Thus ***frequency warping*** exists. It is defined as :

$$\omega = 2 \tan^{-1} (\Omega T_s / 2)$$

- Bilinear Transformation is generally only used in the design of frequency selective filter.
- $T_s$  does not enter into the filter design process because it only scales the  $j\Omega$  axis in the frequency warping function and it can be done in the analog filter specification.
- The prewarping function is needed to determine the passband and cut-off frequencies of analog low-pass filter. It can be done by inverting the equation above:

$$\Omega = 2/T_s \tan (\omega/2)$$



# Filter Conversion : Example using Bilinear

Design the Digital Low-Pass Filter with 3-dB Cut-Off Frequency,  $\omega_c = 0.25\pi$  by using **Bilinear Transformation** method to analog Butterworth Low-Pass Filter defined below:

$$H_a(s) = [1 / 1 + (s/\Omega_s)]$$

## Solution :

1. Use the prewarping function to determine  $\Omega$ .

$$\begin{aligned}\Omega &= 2/T_s \tan (\omega/2) = 2/T_s \tan(0.25\pi/2) \\ &= 0.828/T_s\end{aligned}$$

2. The Analog Filter Function will be :

$$H_a(s) = [1 / 1 + (sT_s/0.828)]$$



# Filter Conversion : Example using Bilinear

3. Now, use Bilinear transformation to obtain  $H(z)$

$$\begin{aligned} H(z) &= H_a(s) \Big|_{s=2/T(1-z^{-1}/1+z^{-1})} \\ &= 1 / (1 + 2/0.828) [(1-z^{-1}) / (1+z^{-1})] \\ &= 0.292 [(1 + z^{-1}) / (1 - 0.4159z^{-1})] \end{aligned}$$

4. The Frequency response of the filter will be :

$$H(e^{j\omega}) = 0.292 [(1 + e^{-j\omega}) / (1 - 0.4159e^{-j\omega})]$$



# Filter Conversion : Example using Bilinear

Determine the Low-Pass Digital Filter with 3-dB Cut-Off Frequency of  $0.2\pi$  and its Frequency Response from the analog filter given below:

$$H_a(s) = \Omega_c / s + \Omega_c$$

where  $\Omega_c$  is the 3-dB Cut-Off Frequency.

## Solution :

1. Use prewarping function to determine  $\Omega$ ,

$$\Omega = 2/T_s \tan (0.2/2) = 0.65/T_s$$

2. The analog transfer function will be:

$$\begin{aligned} H_a(s) &= \Omega_c / s + \Omega_c \\ &= 0.65/T_s / s + 0.65/T_s \end{aligned}$$

$T_s$  has been cancelled out.



# Filter Conversion : Example using Bilinear

3. Now, Use Bilinear Transformation method to obtain,  $H(z)$ ,

$$\begin{aligned} H(z) &= H_a(s) \Big|_{s=2/T(1-z^{-1}/1+z^{-1})} \\ &= 0.245(1+z^{-1}) / (1-0.509z^{-1}) \end{aligned}$$

4. Thus, the frequency response of the filter is :

$$H(e^{j\omega}) = 0.245 (1 + e^{-j\omega}) / (1 - 0.509e^{-j\omega})$$

5. at  $\omega = 0$ ,  $H(0) = 1$ , and at  $\omega = 2\pi$ ,

$$| H(e^{j0.2\pi}) | = 0.707 \text{ – the desired response}$$



# Filter Conversion :

## Example using Impulse Invariance

- Impulse Invariance method is defined as below;

$$H(z) = \sum [A_k / (1 - e^{pkT}) z^{-1}]$$

$A_k$  = Partial Fraction Coefficients

$pk$  = Poles for Transfer Function,  $H(z)$

$T$  = Sampling Interval / Period





# Filter Conversion :

## Example using Impulse Invariance

- Convert the following Analog filter into Digital filter using Impulse Invariance method where the sampling frequency is 20 Hz.

$$H(s) = (6s + 6) / (s^2 + 5s + 6)$$

### Solution :

1. Use PFE technique to the Transfer function above into its Coefficient and Poles.

2.  $H(s) = (6s + 6) / (s + 2)(s + 3) = A_1/(s+2) + A_2/(s+3)$

3. Solve for A1 and A2;  $A_1 = -6$ ,  $A_2 = 12$ , thus;

$H(s) = -6/(s+2) + 12/(s+3)$  where  $P_1 = -2$  and  $P_2 = -3$ ,  $T=1/F = 0.05$

4. Thus,  $H(z) = -6 / (1 - e^{-2(0.05)}) + 12 (1 - e^{-3(0.05)})$



# Filter Conversion : Example using Impulse Invariance

- **Solution :**

5. Thus,  $H(z) = -6 / (1 - e^{-0.1}z^{-1}) + 12 (1 - e^{-0.15}z^{-1})$

6. Simplify the equation;

$$H(z) = -6 / (1 - 0.905z^{-1}) + 12 / (1 - 0.861z^{-1})$$

7. Thus, the final digital transfer using Impulse Invariance method is;

$$H(z) = (6 - 5.694z^{-1}) / (1 - 1.766z^{-1} + 0.8z^{-2})$$



# INTRODUCTION TO DISCRETE-TIME SIGNAL

To design IIR filter using filter specifications and perform filter conversion using Bilinear and Impulse Invariance

5 Techniques

4  
To configure IIR filter based on filter parameters



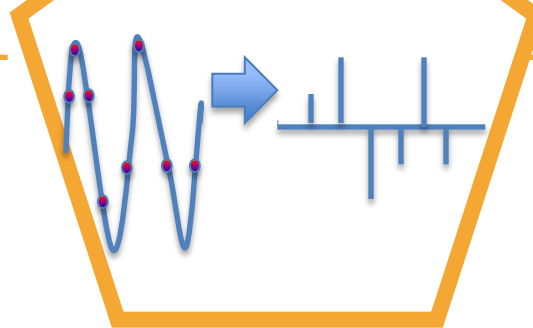
1  
To identify type of IIR filter and filter specifications



3  
To obtain correct IIR filter response



2  
To perform IIR filter conversion

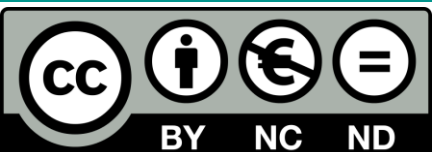


# Conclusion

- Able to differentiate the type and the characteristics of IIR filters.
- Able to design and produce IIR filter response based on the filter specifications.
- Able to perform filter conversion from analog filter to digital filter and vice versa.



**Dr. Norizam Sulaiman,**  
Senior Lecturer,  
Applied Electronics and Computer  
Engineering,  
Faculty of Electrical & Electronics  
Engineering, Universiti Malaysia Pahang,  
Pekan Campus, Pekan, Pahang, Malaysia  
[norizam@ump.edu.my](mailto:norizam@ump.edu.my)



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