DIGITAL SIGNAL PROCESSING

Chapter 4
System Representation

by
Dr. Norizam Sulaiman
Faculty of Electrical & Electronics Engineering
norizam@ump.edu.my

System Representation

- Aims
  - To explain the method of system representation in LTI system, convolution in frequency domain and frequency response.

- Expected Outcomes
  - At the end of this course, students should be able to represent LTI system in using difference equation, transfer function and impulse response.
  - Perform convolution in frequency domain.
  - Find the frequency response of the system.

System Representation

- LTI System can be represented in 3 ways;
  1. Difference Equation, y(n)
  2. Transfer Function, H(z)
  3. Impulse response, h(n)

Difference Equation

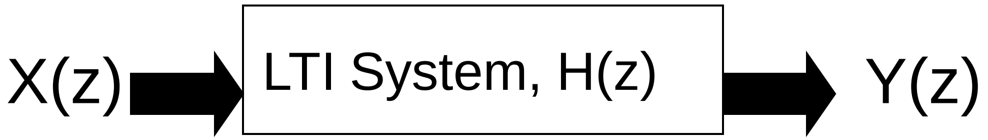
- It is presented in discrete-time domain shown in example below;

➔ **$y(n) = 3y(n-1) + y(n) – 2x(n-1) + 3x(n)$**

- The signal representation should start with y(n) which represent the output of the system. Meanwhile, x(n) stands for the input of the system.
- The type of LTI system (FIR or IIR) can be determined from the system difference equation.

Transfer Function

- The **Transfer Function** is defined as the ratio of the output sequence over input sequence. It is always expressed in term of z-domain as shown below:



$$Y(z) = H(z)X(z)$$

$$H(z) = Y(z) / X(z)$$ =>System Transfer Function

- The **Transfer Function H(z)** is derived from the z-transform of **Impulse Response**, h(n) as shown below:

$$H(z) = \sum_{n=\infty}^{\infty} h(n)z^{-n}$$

Transfer Function

- The transfer function, $H(z)$ is the ratio of 2 polynomials in z^{-1}.

$$H(z) = N(z)/D(z) = \frac{\sum_{k=0}^{M} b_k z^{-k}}{1 + \sum_{k=1}^{N} a_k z^{-k}}$$

$$= \frac{b_0 + b_1 z^{-1} + \dots + b_M z^{-M}}{1 + a_1 z^{-1} + \dots + a_N z^{-N}}$$

- H(z) is obtained from the system described by a linear constant-coefficient difference equation of the form:

$$y(n) = -\sum_{k=1}^{N} a_k y(n-k) + \sum_{k=0}^{M} b_k x(n-k)$$

Transfer Function : Example

Determine the system function and the unit sample response of the system described by the difference equation :

$$y[n] = \tfrac{1}{2}\, y[n-1] + 2x[n]$$

Solution :

1. Convert the equation above into z-transform
   -transform as shown below:

$$Y(z) = \tfrac{1}{2}\, z^{-1}\, Y(z) + 2X(z)$$

$$(1 - \tfrac{1}{2}\, z^{-1})\, Y(z) = 2X(z)$$

2. Now, define system Transfer Function :
   $H(z) = Y(z) / X(z) = 2 / (1 - \tfrac{1}{2}\, z^{-1})$, no zero & pole, $p_1 = 1/2$

3. By using the inverse z-transform, the unit sample or impulse response can be determined:
   $h(n) = Ƶ^{-1}\{H(z)\} = 2(1/2)^n\, u[n]$

Transfer Function : Example

Determine the system function and the unit sample response of the system described by the difference equation :

$$y[n] - y[n-1] = x[n] + x[n-1]$$

Solution :

1. Convert the equation above into z-transform
   -transform as shown below:

$$Y(z) - z^{-1}Y(z) = X(z) + z^{-1}X(z)$$

$$(1 - z^{-1})\,Y(z) = (1 + z^{-1})X(z)$$

2. Now, define system Transfer Function :
   $H(z) = Y(z) / X(z) = (1 + z^{-1}) / (1 - z^{-1})$, zero at -1 & pole at 1

$$= (1 / 1\text{-}z^{-1}) + (z^{-1} / 1\text{-}z^{-1})$$

3. By using the inverse z-transform, the unit sample or impulse response can be determined:
   $h(n) = Ƶ^{-1}\{H(z)\} = u[n] + u[n-1]$

Impulse response

- Impulse response or unit sample response, is a system representation in time-domain, **h(n).**
- It can be obtained from difference equation that used to describe the system.
- There are two types of impulse response;
  - ➔ FIR (Finite Impulse Response) – Non-Recursive
  - ➔ IIR (Infinite Impulse Response) - Recursive

FIR System (Non-recursive)

- It is a system where the output of the system only depend on the input signals.
- The system only has zeros and no poles.
- The system has no feedback.
- The system always stable.
- Example of the difference equation that can describe the system;

➔ **$y(n) = x(n) + x(n-2) - 2x(n+1)$**

IIR System (Recursive)

- It is a system where the output of the system not only depend on the input signals but the past values of the output signals.
- The system only has both zeros and poles.
- The system has feedback.
- The stability of the system depends on its poles.
- Example of the difference equation that can describe the system;

➔ **$y[n] = \frac{1}{2}\, y[n-1] + 2x[n] + x[n-1]$**

Convolution - Example

- In LTI system, the convolution of input signal with the impulse response of the system involved the **finite** and **infinite sequence** of either input signal or impulse response.

- Example:

  Perform convolution to the following 2 finite input signals;

  $x_1(n) = \{1, -2, 1\}$

  $$x_2(n) = \begin{bmatrix} 1, & 0 \leq n \leq 5, \\ 0, & \text{elsewhere} \end{bmatrix}$$

Convolution - Example

$x_1(n) = \{1, -2, 1\} \rightarrow X_1(z) = 1 - 2z^{-1} + z^{-2}$

$x_2(n) = \begin{cases} 1, & 0 \le n \le 5, \\ 0, & \text{elsewhere} \end{cases}$

$X_2(z) = 1 + z^{-1} + z^{-2} + z^{-3} + z^{-4} + z^{-5}$

Perform convolution;

$X(z) = X_1(z)X_2(z) = (1 - 2z^{-1} + z^{-2})(1 + z^{-1} + z^{-2} + z^{-3} + z^{-4} + z^{-5})$

$= (1 - z^{-1} - z^{-6} + z^{-7})$

Perform inverse of X(z) to get x(n) $\rightarrow$ {1, -1, 0, 0, 0, 0, 0, -1, 1}

Convolution - Example

- Example:

  Perform convolution to the following infinite input signals and impulse response of the system;

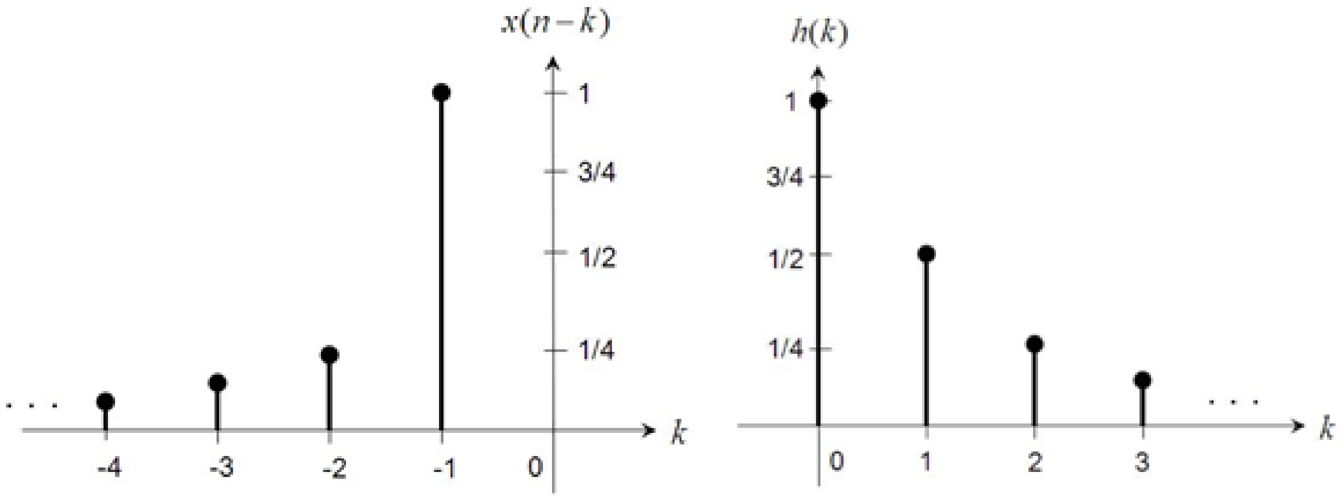
$$h(n) = \left(\frac{1}{2}\right)^n u(n)$$

$$x(n) = \left(\frac{1}{4}\right)^n u(n)$$

Convolution - Example

- Change all n to k and perform signal shifting by k.
- For better view, convert both signal into graphical form as shown below.

- As usual, changed all *n* to *k*
- Apply step 1 of convolution process which is inverting one of the signal (in this case $x(n)$ was chosen)
- Next, apply step 2 where the inverted signal is shifted, $x(n\text{-}k)$



Convolution - Example

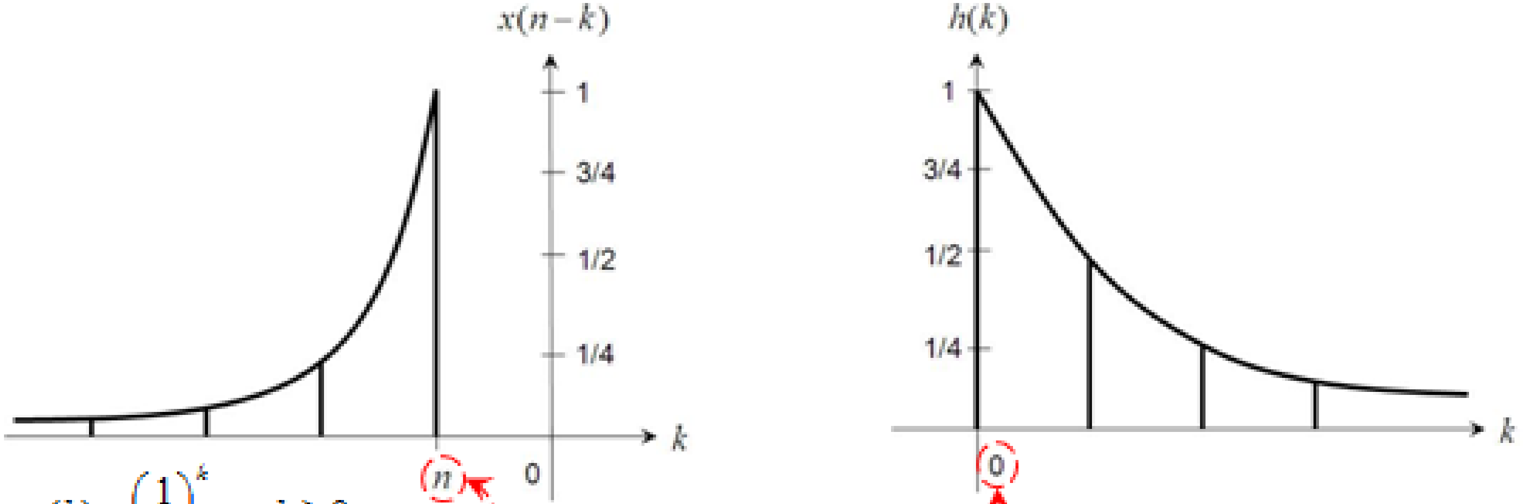
- The direct convolution equation will be:

$$y(n) = h(n) * x(n) = \sum_{k=-\infty}^{\infty} h(k)x(n-k) = \sum_{k=-\infty}^{\infty} \left(\frac{1}{2}\right)^{k} u(k) \left(\frac{1}{4}\right)^{n-k} u(n-k)$$

- For a causal or anticausal signal, the signal will only have the start point but doesn't have any end point
- When convolving, the start point and end point is very crucial as these points will determine where the convolution result is non zero

Convolution - Example

- As for that, both of the signals can be represented as:



$x(k) = \left(\frac{1}{4}\right)^{k}$, $k \geq 0$

$x(-k) = \left(\frac{1}{4}\right)^{-k}$, $k \leq 0$

$x(n-k) = \left(\frac{1}{4}\right)^{n-k}$, $k \leq n$

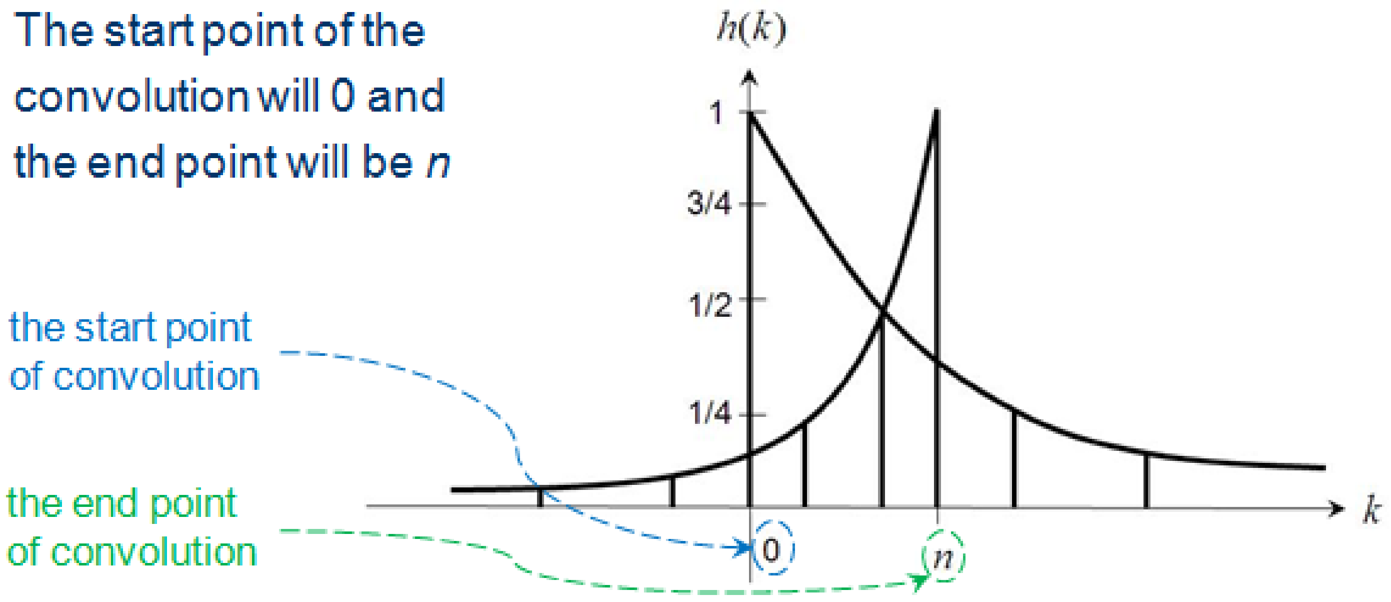
the starting point of $x(n\text{-}k)$

$h(k) = \left(\frac{1}{2}\right)^{k}$, $k \geq 0$

the starting point of $h(k)$

Convolution - Example

- It is clear that for $x(n-k)$ will only have values for $k \le n$ and $h(k)$ will only have values for $k \ge 0$
- When both of the signals is convolved, the result will be:
- The start point of the convolution will 0 and the end point will be n



Convolution - Example

- After determining the convolution's start and end point, the direct convolution equation will be:

$$y(n)=\sum_{k=0}^{n}\left(\frac{1}{2}\right)^{k}\left(\frac{1}{4}\right)^{n-k}=\sum_{k=0}^{n}\left(\frac{1}{2}\right)^{k}\left(\frac{1}{4}\right)^{n}(4)^{k}=\left(\frac{1}{4}\right)^{n}\sum_{k=0}^{n}(2)^{k}$$

- By using closed-form expression where $N = n+1$

$$y(n)=\sum_{n=0}^{N-1}a^{n}=\frac{1-a^{N}}{1-a}=\left(\frac{1}{4}\right)^{n}\sum_{k=0}^{n+1-1}2^{k}=\left(\frac{1}{4}\right)^{n}\frac{1-2^{n+1}}{1-2}\quad,\quad n\geq 0$$

$$=\left(\frac{1}{4}\right)^{n}\left(2^{n+1}-1\right)\quad,\quad n\geq 0$$

Convolution - Example

- Convolution in the LTI system also involved the finite sequence with infinite sequence with the following properties;

$$\delta(n-k) * x(n) = x(n-k)$$

$$\delta(n) * x(n-k) = x(n-k)$$

$$\delta(n-k) * x(n-l) = x(n-k-l)$$

Convolution - Example

- Perform Convolution on the following finite input signal with infinite impulse response.

$$h(n) = -2\left(\frac{1}{2}\right)^{n} u(n)$$

$$x(n) = \frac{1}{2}\delta(n)$$

Convolution - Example

- Perform Convolution on the following finite input signal with infinite impulse response.
  - By using $\delta(n-k) * x(n) = x(n-k)$ where $k = 0$:

$$\delta(n) * x(n) = x(n)$$

$$\frac{1}{2}\delta(n) * \left[-2\left(\frac{1}{2}\right)^n u(n)\right] = -\left(\frac{1}{2}\right)^n u(n)$$

Convolution - Example

- Perform Convolution on the following finite input signal with infinite impulse response.

$$h(n) = -2\left(\frac{1}{2}\right)^n u(n)$$

$$x(n) = -\frac{1}{4}\delta(n-1)$$

Convolution - Example

- By using $\delta(n-k) * x(n) = x(n-k)$ where *k* = 1:

$$\delta(n-1) * x(n) = x(n-1)$$

$$-\frac{1}{4}\delta(n-1) * \left[-2\left(\frac{1}{2}\right)^{n} u(n)\right] = \frac{1}{2}\left(\frac{1}{2}\right)^{n-1} u(n-1)$$

System response - Example

Determine the response of the following system

$$y(n) = -0.1y(n-1) + 0.2y(n-2) + x(n) + x(n-1)$$

if the input signal is

$$x(n) = (1/3)^n u(n)$$

Convolution - Example

- **Solution :**

1. Find the transfer function of the system;

   $H(z) = (1 + z^{-1})/ (1 + 0.1z^{-1} - 0.2 z^{-2})$

2. Factor out the Denominator of H(z);

   $H(z) = (1 + z^{-1}) / (1 - 0.4z^{-1}) (1 + 0.5 z^{-1})$, zero at $z = -1$, poles at $z = 0.4$ and -0.5.

3. Perform z transform to the input signal;

   $X(z) = 1/ (1 - 1/3 z^{-1})$

4. Now, perform convolution;

   $Y(z) = H(z)X(z)$

   $= [(1 + z^{-1}) / (1 - 0.4 z^{-1})(1 + 0.5 z^{-1})] (1 / (1 - 1/3 z^{-1}))$

   $= (1 + z^{-1}) / (1 - 0.4 z^{-1})(1 + 0.5 z^{-1})(1 - 1/3 z^{-1})$

5. Using inverse z-transform by partial fraction technique to obtain **y(n).**

$Y(z) = A / (1 - 0.4 z^{-1}) + B / (1 + 0.5 z^{-1}) + C / (1 - 1/3 z^{-1})$

$A = (1 + z^{-1}) / (1 + 0.5 z^{-1})(1 - 1/3 z^{-1})$ at $z = 0.4$

$= 28/3$

Convolution - Example

$Y(z) = A / (1 – 0.4 z^{-1}) + B / (1 + 0.5 z^{-1}) + C / (1 – 1/3 z^{-1})$

$B = (1 + z^{-1}) / (1 - 0.4 z^{-1})(1 – 1/3 z^{-1})$ at $z = -0.5$

$= -1/3$

$C = (1 + z^{-1}) / (1 - 0.4 z^{-1})(1 + 0.5 z^{-1})$ at $z = 1/3$

$= -8$

Thus,

$Y(z) = (28/3) / (1 – 0.4 z^{-1}) -1/3 / ((1 + 0.5 z^{-1}) – 8 /((1 – 1/3 z^{-1})$

Convert to y(n) by referring to the table of z-transform common pairs;

$$\mathbf{y(n) = [28/3\ (2/5)^n + 1/3\ (-1/2)^n – 8(1/3)^n\ ]u(n)}$$

Frequency Response

- The LTI system can be transformed into frequency domain using **Fourier Transform**.
- To convert the system into its frequency domain, need to start from its z-plane;

  $z = re^{j\omega}$ or $re^{j\theta}$ where r is the radius and ω or θ

  is the angle of the circle. If r =1, $z = e^{j\omega}$ which

  represent the unit circle.
- Thus, by replacing z with $e^{j\omega}$, the LTI system is converted from z-transform to its frequency domain.
- The response of the LTI system after replacing z with

  $e^{j\omega}$ is called **Frequency Response** as shown in

  equation below;

$$H(z) = H(e^{j\omega}) = H(\omega)$$

Frequency Response

- Basically, frequency response of LTI system consists of complex number where the magnitude and phase can be defined as below;

$$|H(e^{j\omega})| = \sqrt{re\{H(\omega)\}^2 + im\{H(\omega)\}^2}$$

$$\phi(\omega) = \tan^{-1} im\{H(\omega)\}/re\{H(\omega)\}$$

- The magnitude and phase can be expressed in polar form as shown below;

$$H(\omega) = |H(\omega)| \angle \phi(\omega)$$

Frequency Response - Example

The LTI system is described by its transfer function below;

$H(z) = 1 / (1 – 0.8z^{-1})$

Calculate the following;

(i) The impulse response, $h(n)$

(ii) Frequency response, $H(\omega)$

(iii) Magnitude and Phase response

Frequency Response - Example

(i) using inverse z-transform and inspection method,

$h(n) = 0.8^n u(n)$

(ii) To get frequency response, replace z of H(z) with $e^{j\omega}$, thus

$H(\omega) = 1 / (1 – 0.8(e^{j\omega})^{-1})$

now, simplify the equation to;

$H(\omega) = 1 / (1 – 0.8e^{-j\omega})$ or $e^{-j\omega} / (e^{-j\omega} – 0.8)$

Frequency Response - Example

(iii) Find Magnitude and phase response using Euler Identity;

$e^{j\omega} = \cos\omega + j\sin\omega$

$e^{-j\omega} = \cos\omega - j\sin\omega$

Thus rewrite H(ω) using Euler's identity,

$H(\omega) = 1 / (1 – 0.8e^{-j\omega})$

$= 1 / (1 – 0.8 (\cos\omega – j\sin\omega))$

$= 1 / (1 – 0.8 \cos\omega + 0.8j\sin\omega)$

$= (\cos\omega + j\sin\omega) / ((\cos\omega + j\sin\omega) – 0.8)$

Then, re-arrange the real and imaginary part for both sides;

Frequency Response - Example

- Solution 22 (continue)

(iii) Magnitude and phase response;

$H(\omega) = (\cos\omega + j\sin\omega) / (\cos\omega - 0.8 + j\sin\omega)$

Then, re-arrange the real and imaginary part for both sides;

$|H(\omega)| =$

$$\frac{1}{\sqrt{1.64 - 1.6 cos\omega}}$$

the actual value of magnitude can be known if ω is replaced with a value. For example, if $\omega = \pi/4$,

$|H(\omega)| = |H(\pi)| = 1 / \sqrt{1.64 - 1.6(0.707)}$

$= 1 / 0.71 =$ **1.4.**

Frequency Response - Example

phase response;

$\phi(\omega) = \tan\text{-}1\ \text{im}\{H(\omega)\}/\text{re}\{H(\omega)\}$

$$= \frac{0^o}{\tan^{-1}[\,(0.8\sin\omega)/(1 - 0.8\cos\omega)}$$

$$= -\tan^{-1}((0.8\sin\omega)/(1 - 0.8\cos\omega))$$

if replace $\omega = \pi/4$,

$\phi(\pi) = -\tan^{-1}((0.5656)\,/\,(0.4343) = 52^o$

- **$H(\omega) = 1.4\angle 52^o$**

SYSTEM REPRESENTATION

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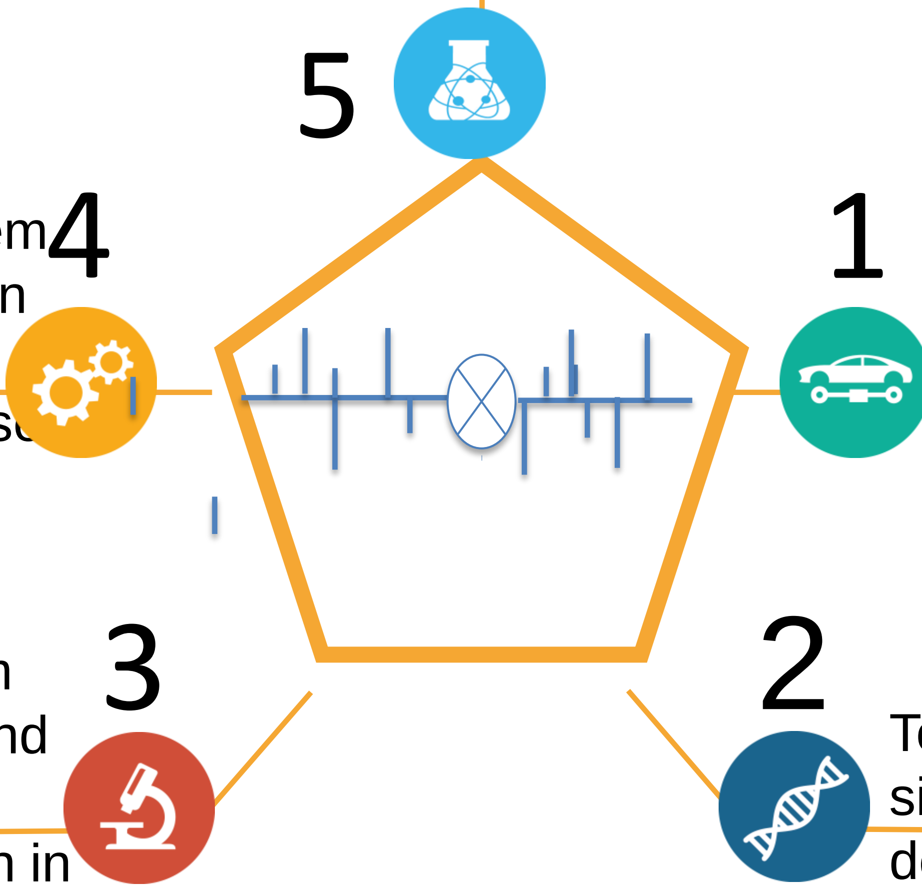
To obtain system representation in term of difference equation, transfer function and impulse response, obtain system response using convolution, and find the frequency response of the system



1 To identify system representation, transfer function, & frequency response

2 To convert the signal into freq. domain and find the response

3 To calculate system Transfer function and impulse response, perform convolution in frequency domain

4 To configure system response based on the convolution & frequency response of the system

5

Conclusion

- Able to apply system representation method to LTI system to obtain transfer function and impulse response of the system.

- Able to perform convolution in frequency domain between discrete-time input signal with its impulse response to obtain system response.

- Able to understand the process to find the frequency response of the system.

Teaching slides prepared by
Dr. Norizam Sulaiman,
Senior Lecturer,
Applied Electronics and Computer Engineering,
Faculty of Electrical & Electronics Engineering, Universiti Malaysia Pahang,
Pekan Campus, Pekan, Pahang, Malaysia