DIGITAL SIGNAL PROCESSING

Chapter 3
z-Transform

by
Dr. Norizam Sulaiman
Faculty of Electrical & Electronics Engineering
norizam@ump.edu.my

Introduction to z-Transform

- Aims
  - To analyze discrete-time signal in the frequency domain using z-transform technique and to perform convolution in frequency domain.

- Expected Outcomes
  - At the end of this course, able to understand how to convert signal from time-domain to frequency domain and perform signal convolution in frequency domain using z-transform technique.

z-Transform: definition

- It is hard to analyze any sampled signal or data in the frequency domain using s-plane (Laplace's Transform). Analysis in z-plane is much preferred.
- The resulting transformation from s-domain to z-domain is called ***z-transform*.**
- The z-transform maps any point $s = \sigma + j\omega$ in the s-plane to z-plane ($r\angle\theta$).
- The relation between s-plane and z-plane and vice-versa is described below :

$$z = e^{sT} \quad , s = (1 / T) \ln z$$

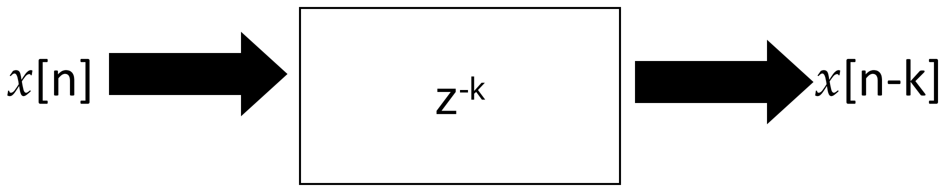
Where T is the sampling period ($T = 1 / F_s$).

z-Transform: definition

- The z-transform of a sequence $x[n]$ is defined as below :

$$Z\{x[n]\} = X(z) = \sum_{n=-\infty}^{\infty} x[n]z^{-n}$$

- From the equation above, it can be seen that z^{-n} correspond to a delay of nT seconds or n sampling period.



- The z-transform can be converted to Fourier Transform (Frequency Domain) by replacing z with $e^{j\omega}$ as shown below :

$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n}$$

z-Transform: definition

- z-transform is used to characterize the LTI system in term of its response to input signal by pole-zero locations.
- z-transform is also used to analyze the characteristic of LTI system in frequency domain while s-transform is used to analyze time-domain signal.

Z-Transform: Region of Convergence (ROC)

- A discrete-time signal is uniquely determined by its z-transform X(z) and the region of the convergence (ROC) of X(z).

- The ROC of the **causal signal** is **exterior** of an unit circle in z-plane. Meanwhile, the ROC of the **anti-causal signal** is **interior** of an unit circle in z-plane.

Z-Transform: ROC Properties

- The properties of ROC are :

1. A finite length sequence has a z-transform with a region of convergence that includes entire z-plane except at $z = 0$ or $z = \infty$.

2. A right sided sequence has a z-transform with a ROC is exterior of the circle:
   ROC : $|z| > \alpha$

3. A left sided sequence has a z-transform with ROC is the interior of a circle.
   ROC : $|z| < \beta$

z-Transform: Table of common z-transform pairs

No	Signal, $x(n)$	Z-Transform	ROC
1	$\delta(n)$	1	All z
2	$u(n)$	$1 / (1 - z^{-1})$	$\lvert z \rvert > 1$
3	$a^n u(n)$	$1 / (1 - az^{-1})$	$\lvert z \rvert > \lvert a \rvert$
4	$na^n u(n)$	$az^{-1} / (1 - az^{-1})^2$	$\lvert z \rvert > \lvert a \rvert$
5	$-a^n u(-n-1)$	$1 / (1 - az^{-1})$	$\lvert z \rvert < \lvert a \rvert$
6	$-na^n u(-n-1)$	$az^{-1} / (1 - az^{-1})^2$	$\lvert z \rvert < \lvert a \rvert$
7	$(\cos\omega_0 n)u(n)$	$(1-z^{-1}\cos\omega_0) / (1-2z^{-1} \cos\omega_0 + z^{-2})$	$\lvert z \rvert > \lvert 1 \rvert$
8	$(\sin\omega_0 n)u(n)$	$(z^{-1}\sin\omega_0) / (1-2z^{-1} \cos\omega_0 + z^{-2})$	$\lvert z \rvert > \lvert 1 \rvert$
9	$a^n(\cos\omega_0 n)u(n)$	$(1-az^{-1}\cos\omega_0) / (1-2az^{-1} \cos\omega_0 + a^2z^{-2})$	$\lvert z \rvert > \lvert a \rvert$
10	$a^n(\sin\omega_0 n)u(n)$	$(az^{-1}\sin\omega_0) / (1-2az^{-1} \cos\omega_0 + a^2z^{-2})$	$\lvert z \rvert > \lvert a \rvert$

z-Transform: Table of z-transform properties

No.	Properties	Time-Domain	Z-Domain	ROC
1	Notation	$x(n)$	$X(z)$	$r_2 < \lvert z \rvert < r_1$
2	Linearity	$a_1x_1(n) + a_2x_2(n)$	$a_1X_1(z) + a_2X_2(z)$	At least the intersection of ROC_1 and ROC_2
3	Time shifting	$x(n-k)$	$Z^{-k}X(z)$	That of X(z) except z=0 if k>0 and $z=\propto$ if k<0
4	Scaling in z-domain	$a^n x(n)$	$X(a^{-1}z)$	$\lvert a \rvert r_2 < \lvert z \rvert < \lvert a \rvert r_1$
5	Time Reversal	$x(-n)$	$X(z^{-1})$	$1/r_1 < \lvert z \rvert < 1/r_2$
6	Conjugate	$x^*(n)$	$X^*(z^*)$	ROC
7	Real	$Re\{x(n)\}$	$½ [X(z) + X^*(z^*)]$	Include ROC
8	Imaginary	$Im\{x(n)\}$	$½ [X(z) - X^*(z^*)]$	Include ROC
9	Differentiation in z-transform	$nx(n)$	$-zdX(z)/dz$	$r_2 < \lvert z \rvert < r_1$
10	Convolution	$x_1(n) * x_2(n)$	$X_1(z) X_2(z)$	At the least the intersection of ROC_1 and ROC_2

z-Transform-Examples

- Examples of z-transform using z-transform formula;

  Given the finite length sequence of discrete-time signal as below;

  $x[n] = \{1, 2, 5, 7, 0, 1\}$

  Determine the z-transform of the sequence and its ROC.

  Solution :

$$Ƶ\{x[n]\} = X(z) = \sum_{n=0}^{\infty}\{1,2,5,7,0,1\}z^{-n}$$

$$= 1.z^{-0} + 2.z^{-1} + 5.z^{-2} + 7.z^{-3} + 0.z^{-4} + 1.z^{-5}$$

$$= \mathbf{1 + 2z^{-1} + 5z^{-2} + 7z^{-3} + 0 + z^{-5}}$$

The ROC is the entire z-plane except z = 0.

z-Transform-Examples

- Examples of z-transform using table of common z-transform pairs;

  Given the discrete-time signal as below;

  $x(n) = 0.8^n u(n)$

  Determine the z-transform of the sequence and its ROC.

  Solution :

  First, inspect the table and find the matching equation in term of z-transforrm.

  The equation match with equation no. 3 in the table where

  a = 0.8, thus,

  $$X(z) = 1 / (1 - az^{-1}) = 1 / (1 - 0.8z^{-1})$$

  The ROC is at IzI > 0.8

Inverse z-Transform

- The inverse of z-transform of the discrete time signal is required in the signal processing in order to convert the analysis of the discrete-time signal in the z-domain to the time domain for signal re-construction.
- The inverse z-transform generate the discrete sequence, *x*[n] from its z-transform, X(z). It can be defined as below :

$$x[n] = \mathcal{Z}^{-1}\{X(z)\}$$

- The z-transform, X(z) is often expressed in the ratio of 2 polynomials in z^{-1} or z (Numerator over Denominator) as shown below , where M is the highest order of numerator and N is the highest order of denominator:

- $$X(z) = N(z)/D(z) = \frac{\sum_{k=0}^{M} b_k z^{-k}}{\sum_{k=0}^{N} a_k z^{-k}}$$

$$= \frac{b_0 + b_1 z^{-1} + \ldots + b_M z^{-M}}{a_0 + a_1 z^{-1} + \ldots + a_N z^{-N}}$$

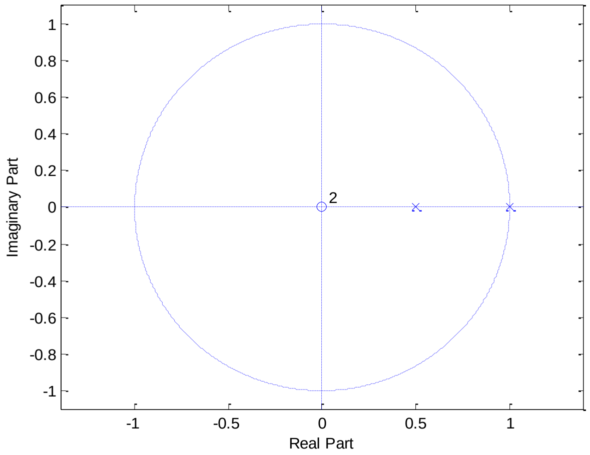
Inverse z-Transform

- The X(z) also can be expressed in a factor form in term of poles and zeros as shown below :

$$X(z) = b_0 z^{-M+N} \frac{(z-z_1)(z-z_2)(z-z_3)\ldots(z-z_M)}{(z-p_1)(z-p_2)(z-p_3)\ldots(z-p_N)}$$

- In the z-plane, the zeros are denoted by "o" and the poles are denoted by "x" as shown in the diagram for the following equation of X(z);

$$X(z) = \frac{1}{1 - 1.5z^{-1} + 0.5z^{-2}}$$



Inverse z-Transform

- There are 3 methods to obtain the inverse z-transform;
  1. Inspection Method
  2. Power Series Expansion Method
  3. Partial Fraction Expansion Method
  4. Residue Method

Inverse z-Transform - Example

- **1. Inspection method**

  Inspect the given z-transform, X(z) and by using the table of common z-transform pairs to determine the inverse of X(z).

  Given the z-transform, $X(z) = (1/1\text{-}0.5z^{-1})$, find the inverse z-transform of X(z).

  Solution :

  First, inspect the table of common z-transform pairs to find the matching one.

  Second, the z-transform is match to $X(z) = 1/1\text{-}\alpha z^{-1}$. In this case, $\alpha = 0.5$.

  Third, write the given z-transform in time domain, which is $X(z) \Rightarrow (0.5)^n u[n]$.

Inverse z-Transform - Example

- **2. Power Series Expansion method**
  Expand the given z-transform X(z) into a power series of the form by a long division.

Example :

$$X(z) = \frac{1}{1 - 1.5z^{-1} + 0.5z^{-2}}$$

- After long division, the z-transform,

$X(z) = 2z^{-2} + 6z^{3} + 14z^{4} + 30z^{5} + 62z^{6} + \ldots$

Thus,

$x[n] = \mathcal{Z}^{-1}\{X(z)\} = \{\ldots, 62, 30, 14, 6, 2, 0, 0\}$

Inverse z-Transform - Example

- **3. Partial Fraction Inspection method**

  In this method, the z-transform is first, expanded into partial fractions. Then, determine the inverse z -transform of each partial fraction from z-transform table and sum them up. If the order of the Numerator less than Denominator (M < N), there is no B_o in front of the partial fraction. If the order of the Numerator = the order of Denominator (M ≥ N), the B_o in front of the partial fraction will be $B_o = b_M / a_N$. The rule of order :

  => If M < N,

  $$X(z)/z = \sum_{k=1}^{N} \frac{A_k z}{z-p_k} = \frac{A_1}{z-p_1} + \frac{A_2}{z-p_2} + \ldots + \frac{A_N}{z-p_N}$$

  => If M = N,

  $$X(z)/z = B_o + \sum_{k=1}^{N} \frac{A_k z}{z-p_k} = B_o + \frac{A_1}{z-p_1} + \frac{A_2}{z-p_2} + \ldots + \frac{A_N}{z-p_N}$$

  => if M > N, then the Numerator must first divided by Denominator through long division to make M ≤ N.

Inverse z-Transform - Example

- **3. Partial Fraction Inspection method**

  Find the discrete-time signal, $x[n]$ represented by the following z-transform by using Partial Expansion Method:

$$X(z) = \frac{1}{1 - 1.5z^{-1} + 0.5z^{-2}}$$

Solution:

1. Express the X(z) in power of z by multiplying the equation by the highest power of z,namely,z^2.
2. The X(z) now become:

$$X(z) = \frac{z^2}{z^2 - 1.5z + 0.5}$$

3. Factor out the denominator by using this formula:

$$P_1,P_2 = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

4. Thus, by using the formula above,

$$P_1,P_2 = \frac{-(-1.5) \pm \sqrt{(-1.5)^2 - 4(1)(0.5)}}{2(1)} = \frac{1.5 \pm 0.5}{2} = 1, 0.5$$

5. Express X(z) as below :

$$\frac{X(z)}{z} = \frac{z}{(z - 1)(z - 0.5)}$$

Inverse z-Transform - Example

- **3. Partial Fraction Inspection method**

Solution:

6. By using PFE, X(z)/z become:

$$\frac{X(z)}{z} = \frac{A_1}{(z-1)} + \frac{A_2}{(z-0.5)}$$

7. Calculate A_1 and A_2 as below :

$$A_1 = \frac{X(z)}{z}(z-1) = \frac{z}{(z-1)(z-0.5)}(z-1)$$

$$= \left.\frac{z}{(z-0.5)}\right|_{z=1} = \frac{1}{(1-0.5)} = 2$$

$$A_2 = \frac{X(z)}{z}(z-0.5) = \frac{z}{(z-1)(z-0.5)}(z-0.5)$$

$$= \left.\frac{z}{(z-1)}\right|_{z=0.5} = \frac{0.5}{(0.5-1)} = -1$$

8. $$X(z) = \frac{2z}{(z-1)} - \frac{z}{(z-0.5)}$$

9. From the table of z-transform, $x[n] = [2 - (0.5)^n]\, u[n]$, $n > 0$.

Inverse z-Transform - Example

- **4. Residue method**

  In this method, the inverse z-transform is obtained by evaluating the contour integral :

$$x[n] = \frac{1}{2\pi j} \oint z^{n-1} X(z)dz$$

$$= \text{sum of the residue of } z^{n-1} X(z) \text{ at all poles inside contour C.}$$

The residue of $z^{n-1} X(z)$ at pole p_k is given by:

$$\text{Res}[F(z), p_k] = \frac{1}{(N-1)!} \frac{d^{N-1}}{dz^{N-1}} [(z - p_k) F(z)]_{z=pk}$$

Inverse z-Transform - Example

- **4. Residue method**

Find the discrete-time sequence, $x[n]$ of the following z-transform by using Residue Method,

$$X(z) = \frac{z^{-1}}{1 - 1.5z^{-1} - 0.5z^{-2}}$$

Solution :

1. The X(z) can be expressed in the form of:

$$X(z) = \frac{z^2}{(z - 1)(z - 0.5)}$$

2. By using Residue Method, let the function $F(z) = z^{n-1} X(z)$

3. F(z) now become :

$$F(Z) = z^{n-1} \frac{z^2}{(z - 1)(z - 0.5)} = \frac{z^{n+1}}{(z - 1)(z - 0.5)} = \frac{z^n z}{(z - 1)(z - 0.5)}$$

4. let n = 0, then F(z) become :

$$F(Z) = \frac{z}{(z - 1)(z - 0.5)}$$

Inverse z-Transform - Example

- **4. Residue method**

Find the discrete-time sequence, x[n] of the following z-transform by using Residue Method,

$$X(z) = \frac{z^{-1}}{1 - 1.5z^{-1} - 0.5z^{-2}}$$

Solution :

5. Now, calculate the residue :

a. $\text{Res}[F(z), 1] = (z-1)\,F(z) = \left.\frac{(z-1)\quad z}{(z-1)(z-0.5)}\right|_{z=1} = \frac{1}{1-0.5} = 2\,(1)^n$

b. $\text{Res}[F(z), 0.5)] = (z-0.5)F(z) = \left.\frac{(z-0.5)\quad z}{(z-1)\,(z-0.5)}\right|_{z=0.5} \quad \frac{0.5}{(0.5-1)} = -1(0.5)^n$

6. Thus the inverse z-transform;

x[n] = [2(1)^n - (0.5)^n] u[n] , n > 0

System Stability & Causality

- **Causal:**
  - ➔ ROC of the LTI system is **exterior of a circle**.
- **Non-Causal**:
  - ➔ ROC of the LTI system in **interior of a circle**.
- **Stable:**

➔The **poles** of the transfer function H(z) **inside the unit circle. ROC must include unit circle.**

- **Unstable:**

➔The **poles** of the transfer function H(z) **outside the unit circle. ROC does not include unit circle.**

System Stability & Causality

The LTI system is described by its transfer function below;

$H(z) = (3 – 4z^{-1}) / (1 – 3.5z^{-1} + 1.5z^{-2})$

First, locate the poles of the system;

$H(z) = (3 – 4z^{-1}) / (1 – 0.5z^{-1})(1 – 3z^{-1})$

The poles at z = 0.5 and 3.
Second, the determine the system stability based on the location of z in the unit circle.
Based on the location of the poles in unit circle, the system is **unstable** since one of the pole located outside unit circle. The system is **non-causal** if ROC is 0.5 <|z| <3
interior and, **causal** if ROC is at |z| > 3, **exterior.**

INTRODUCTION TO Z-TRANSFORM

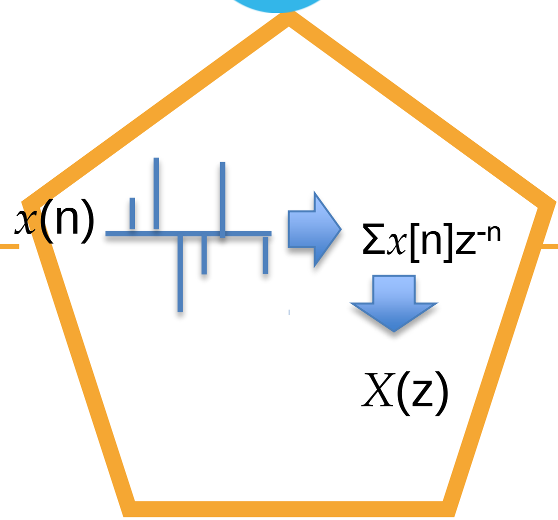
1 To identify z-plane & s-plane

2 To convert the discrete-time signal to frequency domain and vice-versa using z-transform

3 To calculate z-transform and inverse z-transform. Transfer function and convolution in using z-transform

4 To improve sampling & convolution process based on signal characteristics

5 To perform signal analysis in z-transform domain and to convert the signal for signal re-construction



Conclusion of The Chapter

- Able to understand and apply the z-transform technique to perform signal analysis in frequency domain.
- Able to apply inverse z-transform technique for signal re-construction.
- Able to calculate system stability and causality
- Able to obtain zero-pole location of the system from its z-plane or unit circle.

Teaching slides prepared by
Dr. Norizam Sulaiman,
Senior Lecturer,
Applied Electronics and Computer Engineering,
Faculty of Electrical & Electronics Engineering, Universiti Malaysia Pahang,
Pekan Campus, Pekan, Pahang, Malaysia