BEE2143 – Signals & Networks

Chapter 3 – Fourier Series

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Trigonometric Fourier Series

Exponential Fourier Series

Symmetry considerations in Fourier Series

Amplitude and phase spectra in Fourier Series

References

Trigonometric Fourier Series

- A Fourier series is an expansion of a periodic function $f(t)$ in terms of an infinite sum of cosines and sines
- In other words, any periodic function can be resolved as a summation of constant value and cosine and sine functions

$$f(t) = \underbrace{a_0}_{\text{dc}} + \underbrace{\sum_{n=1}^{\infty}(a_n \cos n\omega_0 t + b_n \sin n\omega_0 t)}_{\text{ac}}$$

- The computation and study of Fourier series is known as harmonic analysis
- This is extremely useful as a way to break up an arbitrary periodic function into a set of simple terms that
- These terms can be plugged in, solved individually, and then recombined to obtain the solution to the original problem or an approximation to it

- Fourier series coefficients:

$$a_0 = \frac{1}{T}\int_0^T f(t)dt$$

$$a_n = \frac{2}{T}\int_0^T f(t)\cos n\omega_0 t dt$$

$$b_n = \frac{2}{T}\int_0^T f(t)\sin n\omega_0 t dt$$

where

$$\omega_0 = \frac{2\pi}{T}$$

- Since $f(t)$ is periodic, it may be more convenient to carry the integrations above from $-T/2$ to $T/2$ or generally from t_0 to $t_0 + T$ instead of 0 to T

- Note that

$$
\begin{aligned}
a_0 =& \frac{1}{T}\int_0^T f(t)dt \\
=& \frac{1}{\text{Period}} \times \text{Area below graph over one period} \\
=& \text{Average value of } f(t) \\
=& \text{dc component of } f(t)
\end{aligned}
$$

- The form

$$f(t) = a_0 + \sum_{n=1}^{\infty} (a_n \cos n\omega_0 t + b_n \sin n\omega_0 t) \qquad (1)$$

is known as the sine-cosine form

- An alternative form of (1) is the amplitude-phase form

$$f(t) = a_0 + \sum_{n=1}^{\infty} A_n \cos(n\omega_0 t + \phi_n)$$

where

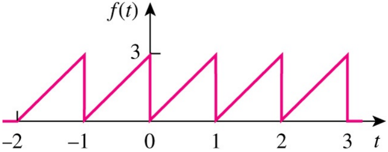
$$A_n = \sqrt{a_n^2 + b_n^2} \quad , \quad \phi_n = -\tan^{-1} \frac{b_n}{a_n}$$

or

$$A_n \underline{/\phi_n} = a_n - jb_n$$

Practice Problem 17.2, pg. 764 (Alexander & Sadiku, 2009)

Determine the Fourier series of the sawtooth waveform in Fig. 17.9.

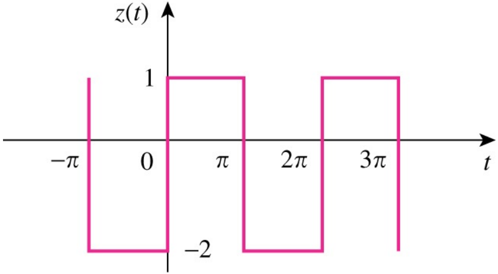


Answer:

$$f(t) = \frac{3}{2} - \sum_{n=1}^{\infty} \frac{3}{n\pi} \sin 2n\pi t$$

Problem 17.5, pg. 799 (Alexander & Sadiku, 2009)

Obtain the Fourier series expansion for the waveform shown in Fig. 17.49.

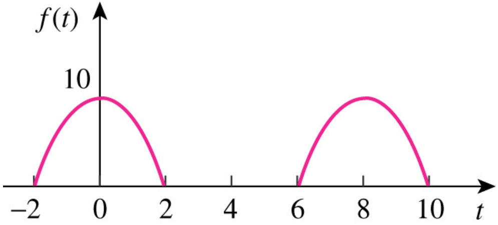


Answer:

$$z(t) = -0.5 + \sum_{\substack{n=1 \\ n=\text{odd}}}^{\infty} \frac{6}{n\pi} \sin nt$$

Problem 17.9, pg. 799 (Alexander & Sadiku, 2009)

Determine the Fourier coefficients a_n and b_n of the first three harmonic terms of the rectified cosine wave in Fig 17.52.



Answer:

$$a_0 = 3.183, \;\; a_1 = 10, \;\; a_2 = 6.362, \;\; a_3 = b_1 = b_2 = b_3 = 0$$

Exponential Fourier Series

- A compact way of expressing the Fourier series in (1) is to put it in exponential form
- This requires that we represent the sine and cosine functions in the exponential form using Euler's identity:

$$
\begin{aligned}
e^{jx} &= \cos x + j\sin x \\
\cos x &= \frac{e^{jx} + e^{-jx}}{2} \\
\sin x &= \frac{e^{jx} - e^{-jx}}{j2}
\end{aligned}
$$

- The complex or exponential Fourier series of $f(t)$ is

$$f(t) = \sum_{n=-\infty}^{\infty} c_n e^{jn\omega_0 t}$$

where

$$c_n = \frac{1}{T}\int_0^T f(t)e^{-jn\omega_0 t}dt$$

- The coefficients of the three forms of Fourier series (sine-cosine form, amplitude-phase form, and exponential form) are related by

$$A_n \underline{/\phi_n} = a_n - jb_n = 2c_n$$

- Note that

$$c_{-n} = c_n^*$$

Example 17.10, pg. 784 (Alexander & Sadiku, 2009)

Find the exponential Fourier series expansion of the periodic function

$$f(t) = e^t, \quad 0 < t < 2\pi$$

with $f(t + 2\pi) = f(t)$.

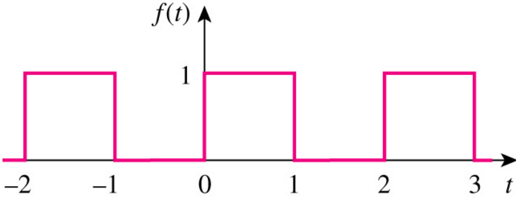
Answer:

$$f(t) = \sum_{n=-\infty}^{\infty} \frac{85}{1 - jn} e^{jnt}$$

$$|c_n| = \frac{85}{\sqrt{1 + n^2}}, \quad \theta_n = \tan^{-1} n$$

Practice Problem 17.10, pg. 785 (Alexander & Sadiku, 2009)

Obtain the complex Fourier series of the function in Fig. 17.1.



Practice Problem 17.10, pg. 785 (cont.)

Answer:

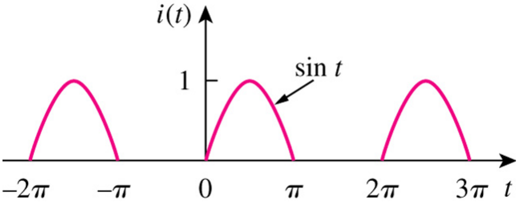
$$f(t) = \frac{1}{2} - \sum_{\substack{n=-\infty \\ n\neq 0 \\ n=\text{odd}}}^{\infty} \frac{j}{n\pi} e^{jn\pi t}$$

*L'Hopital's rule: If $f(0) = 0 = g(0)$, then

$$\lim_{x\to 0} \frac{f(x)}{g(x)} = \lim_{x\to 0} \frac{\frac{d}{dx} f(x)}{\frac{d}{dx} g(x)}$$

Problem 17.55, pg. 805 (Alexander & Sadiku, 2009)

Obtain the exponential Fourier series expansion of the half-wave rectified sinusoidal current of Fig. 17.82.



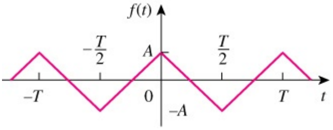
Answer:

$$f(t) = \sum_{n=-\infty}^{\infty} \frac{1 + e^{-jn\pi}}{2\pi(1 - n^2)} e^{jnt}$$

Symmetry considerations in Fourier Series

- Symmetry functions:
  - even symmetry
  - odd symmetry
  - half-wave symmetry
- Any function $f(t)$ is even if its plot is symmetrical about the vertical axis, i.e.

$$f(-t) = f(t)$$



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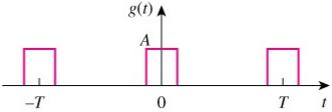
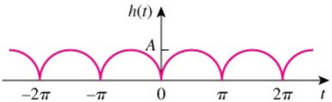
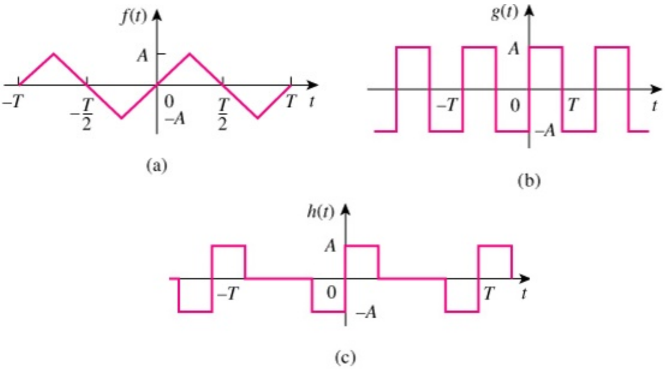


(b)



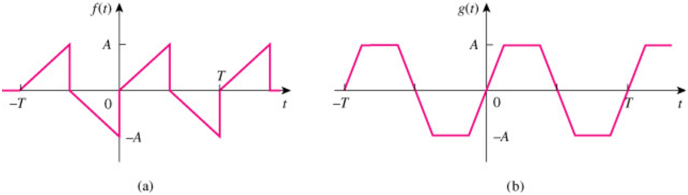
- Any function $f(t)$ is odd if its plot is antisymmetrical about the vertical axis, i.e.

$$f(-t) = -f(t)$$



- A function $f(t)$ is half-wave (odd) symmetric if

$$f\left(t - \frac{T}{2}\right) = -f(t)$$



- The product properties of even and odd functions are:
  - (even)×(even)= (even)
  - (odd)×(odd)= (even)
  - (even)×(odd)= (odd)
  - (odd)×(even)= (odd)
- The integrals properties of even and odd functions are:

$$\int_{-T/2}^{T/2} f_{\text{even}}(t)dt = 2\int_{0}^{T/2} f_{\text{even}}(t)dt$$

$$\int_{-T/2}^{T/2} f_{\text{odd}}(t)dt = 0$$

- Utilizing this property, the Fourier coefficients for an even function become

$$\begin{aligned} a_0 &= \frac{2}{T}\int_0^{T/2} f(t)dt \\ a_n &= \frac{4}{T}\int_0^{T/2} f(t)\cos n\omega_0 t dt \\ b_n &= 0 \end{aligned}$$

- And the Fourier coefficients for an odd function become

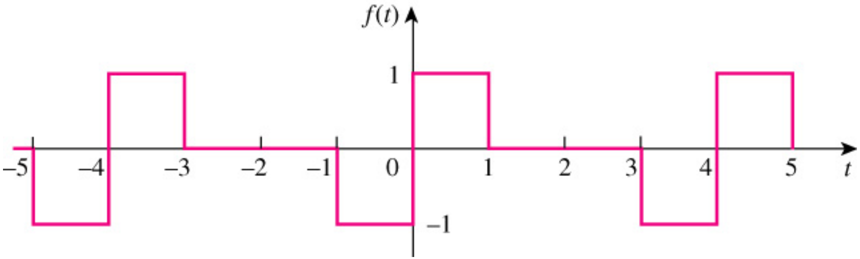
$$\begin{aligned} a_0 = a_n &= 0 \\ b_n &= \frac{4}{T}\int_0^{T/2} f(t)\sin n\omega_0 t dt \end{aligned}$$

- For a half-wave symmetry function, its Fourier coefficients are

$$
\begin{aligned}
a_0 &= 0 \\
a_n &= \begin{cases} \frac{4}{T}\int_0^{T/2} f(t)\cos n\omega_0 t dt, & n \text{ odd} \\ 0, & n \text{ even} \end{cases} \\
b_n &= \begin{cases} \frac{4}{T}\int_0^{T/2} f(t)\sin n\omega_0 t dt, & n \text{ odd} \\ 0, & n \text{ even} \end{cases}
\end{aligned}
$$

Example 17.3, pg. 771 (Alexander & Sadiku, 2009)

Find the Fourier series expansion of $f(t)$ given in Fig. 17.13.

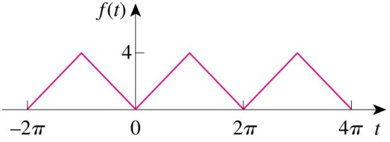


Answer:

$$f(t) = \sum_{n=1}^{\infty} \frac{2}{n\pi}\left(1 - \cos\frac{n\pi}{2}\right)\sin\frac{n\pi t}{2}$$

Practice problem 17.4, pg. 773 (Alexander & Sadiku, 2009)

Find the Fourier series expansion of the function in Fig. 17.16.

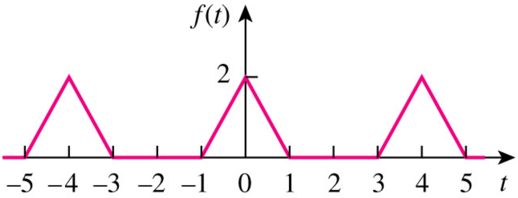


Answer:

$$f(t) = 2 - \sum_{\substack{n=1 \\ n=\text{odd}}}^{\infty} \frac{16}{n^2\pi^2} \cos nt$$

Problem 17.21, pg. 800 (Alexander & Sadiku, 2009)

Determine the trigonometric Fourier series of the signal in Fig. 17.59.



Answer:

$$f(t) = \frac{1}{2} + \sum_{n=1}^{\infty} \frac{8}{n^2\pi^2}\left[1 - \cos\left(\frac{n\pi}{2}\right)\right]\cos\left(\frac{n\pi t}{2}\right)$$

Amplitude and phase spectra in Fourier Series

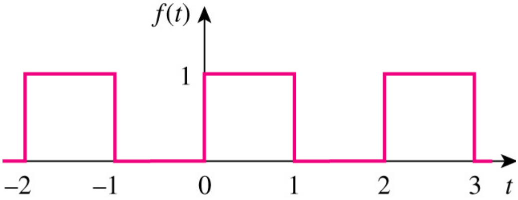
- The plot of the amplitude A_n of the harmonics versus $n\omega_0$ is called the amplitude spectrum of $f(t)$; the plot of the phase ϕ_n versus $n\omega_0$ is the phase spectrum of $f(t)$; both of them form the frequency spectrum of $f(t)$
- Similarly, the plots of the magnitude and phase of c_n versus $n\omega_0$ are called the complex amplitude spectrum and complex phase spectrum of $f(t)$, respectively; both of them form the complex frequency spectrum of $f(t)$
- They are related by

$$c_0 = a_0, \quad |c_n| = |c_{-n}| = \frac{A_n}{2}$$

$$\theta_0 = \phi_0 = 0, \quad \theta_n = \phi_n, \quad \theta_{-n} = -\phi_n$$

Example 17.1, pg. 760 (Alexander & Sadiku, 2009)

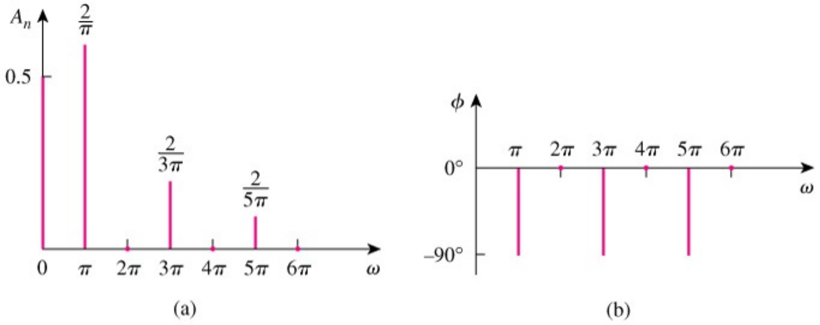
Determine the Fourier series of the waveform shown in Fig. 17.1. Obtain the amplitude and phase spectra.



Example 17.1, pg. 760 (cont.)

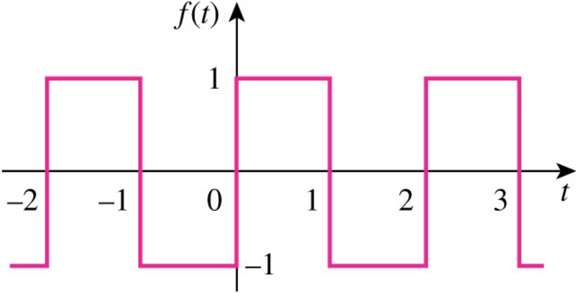
Answer:

$$f(t) = \frac{1}{2} + \sum_{\substack{n=1 \\ n=\text{odd}}}^{\infty} \frac{2}{n\pi} \sin n\pi t$$



Practice Problem 17.1, pg. 762 (Alexander & Sadiku, 2009)

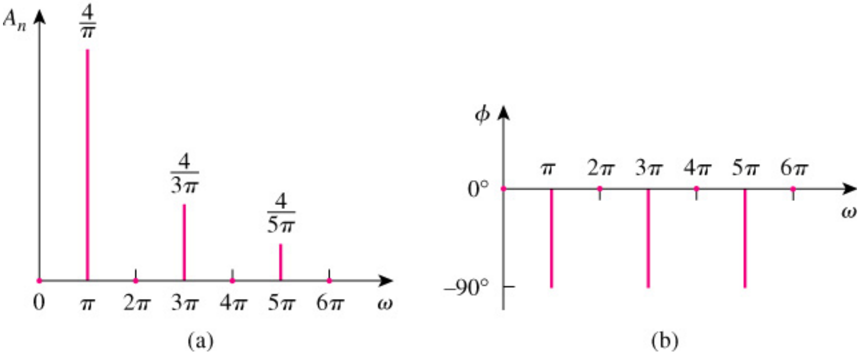
Find the Fourier series of the square wave in Fig. 17.5. Plot the amplitude and phase spectra.



Practice Problem 17.1, pg. 762 (cont.)

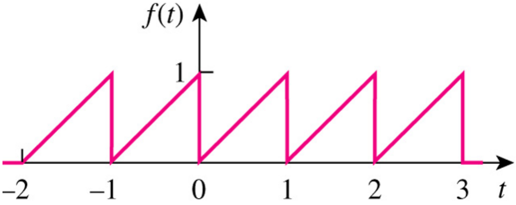
Answer:

$$f(t) = \sum_{\substack{n=1 \\ n=\text{odd}}}^{\infty} \frac{4}{n\pi} \sin n\pi t$$



Example 17.11, pg. 785 (Alexander & Sadiku, 2009)

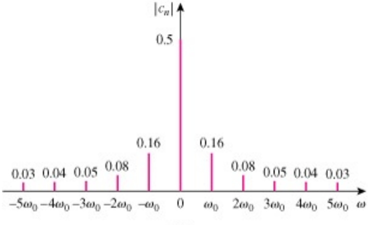
Find the complex Fourier series of the sawtooth wave in Fig. 17.9 (corrected). Plot the amplitude and the phase spectra.



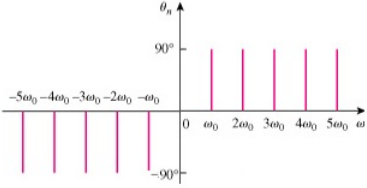
Example 17.11, pg. 785 (cont.)

Answer:

$$f(t) = 0.5 + \sum_{\substack{n=-\infty \\ n\neq 0}}^{\infty} \frac{j}{2n\pi} e^{j2n\pi t}$$



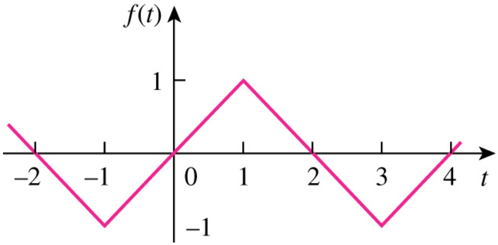
(a)



(b)

Practice Problem 17.11, pg. 787 (Alexander & Sadiku, 2009)

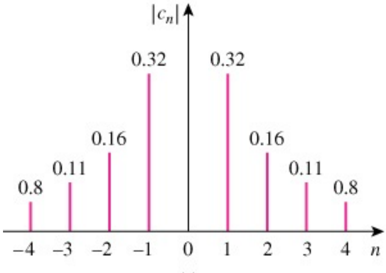
Obtain the complex Fourier series expansion of $f(t)$ in Fig. 17.17. Show the amplitude and phase spectra.



Practice Problem 17.11, pg. 787 (cont.)

Answer:

$$f(t) = -\sum_{\substack{n=-\infty \\ n\neq 0}}^{\infty} \frac{j(-1)^n}{n\pi} e^{jn\pi t}$$



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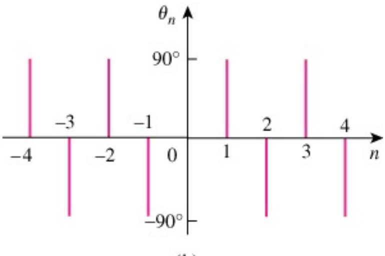

(b)

List of References

1. C.K. Alexander and M.N.O. Sadiku (2009), Fundamentals of Electric Circuits 4th ed., New York: McGraw-Hill.