Universiti Malaysia PAHANG
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BMM4893: Mechanics of Composite Materials

Chapter 3: Macromechanical Analysis of a Lamina

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Synopsis

This chapter explains the stress-strain relationship for different type of materials, develop stress-strain relationship and analyse the engineering constant for an unidirectional lamina. Later, the student will discussing the stress-strain relationship, elastic moduli and strengths of an angle ply based on an unidirectional lamina and the angle of the ply.

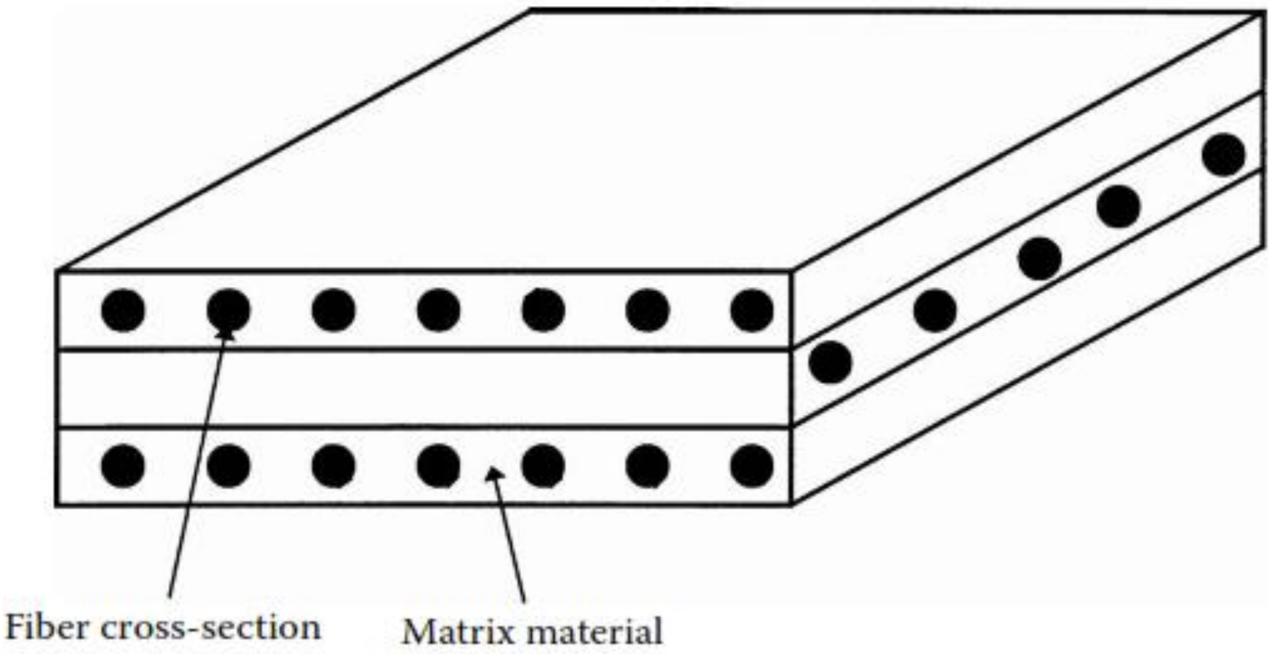
Learning Outcome

By the end of this topic, student should be able to:

- Develop stress-strain relationship for different type of materials
- Develop stress-strain relationship for an unidirectional lamina
- Analyse the engineering constant for an unidirectional lamina
- Develop stress-strain relationship, elastic moduli and strengths of an angle ply based on an unidirectional lamina and the angle of the ply

Introduction

Typical laminate made of three lamina



Introduction

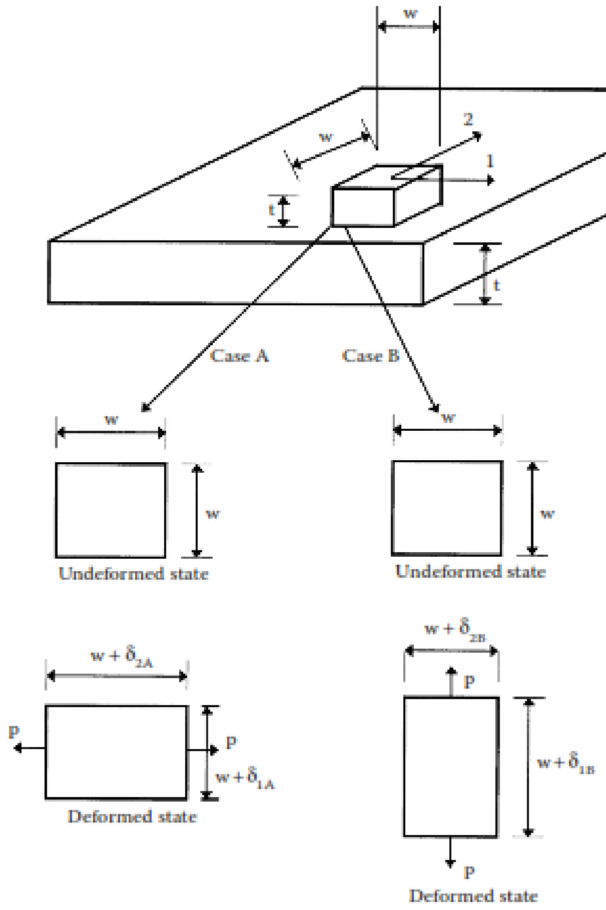
Homogenization of a lamina vs Homogeneous isotropic material

Case A: Subject the square plate to a pure normal load P in direction 1. Measure the normal deformations in directions 1 and 2, δ_{1A} and δ_{2A}, respectively.

Case B: Apply the same pure normal load P as in case A, but now in direction 2. Measure the normal deformations in directions 1 and 2, δ_{1B} and δ_{2B}, respectively.

Introduction

Metal – isotropic plate to pure normal load in direction 1

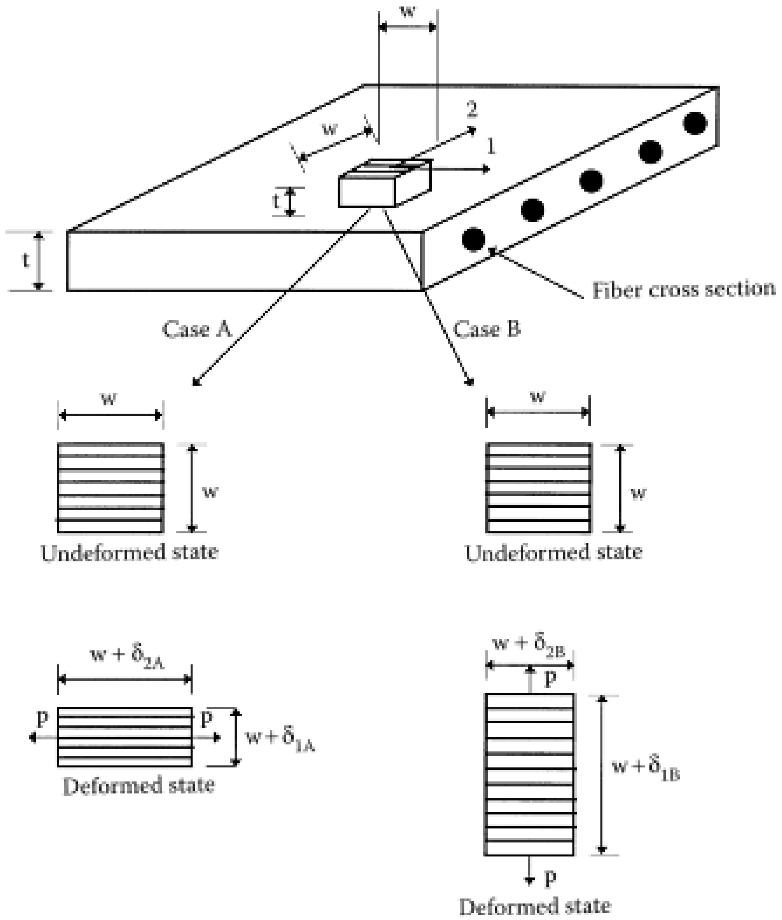


$$\delta_{1A} = \delta_{2B} \,,$$

$$\delta_{2A} = \delta_{1B} \,.$$

Introduction

Unidirectional lamina

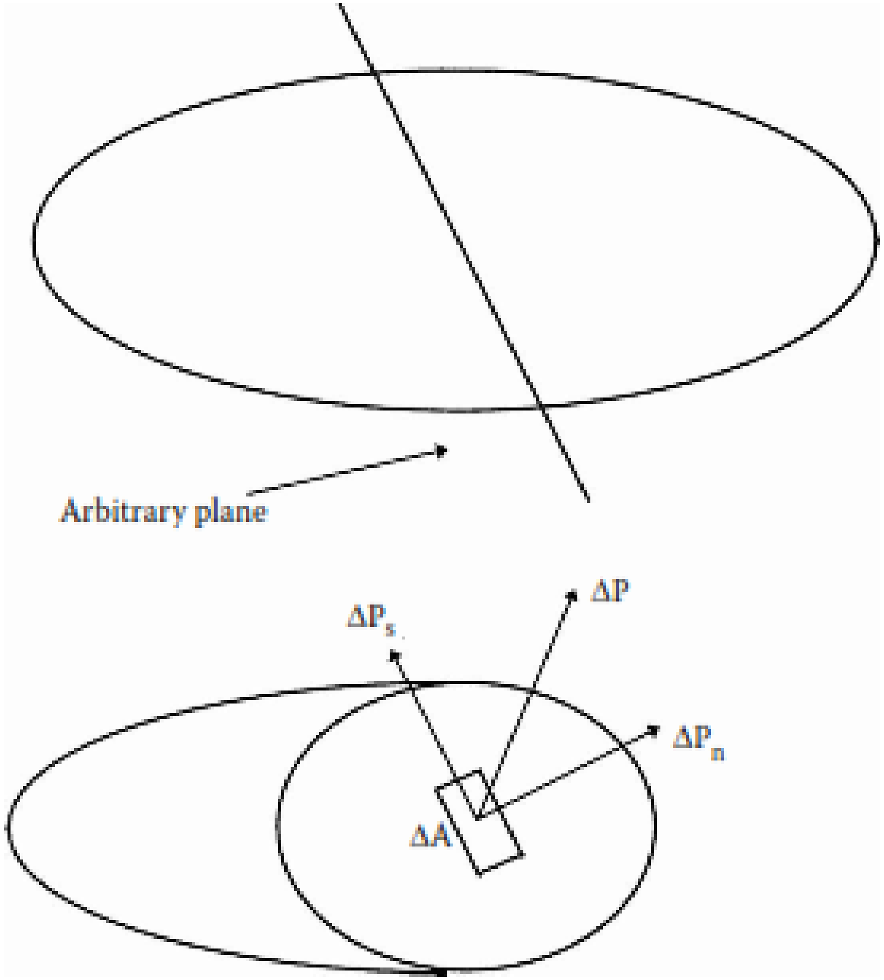


$$\delta_{1A} \neq \delta_{2B}\,,$$

$$\delta_{2A} \neq \delta_{1B}\,.$$

Review

Stress

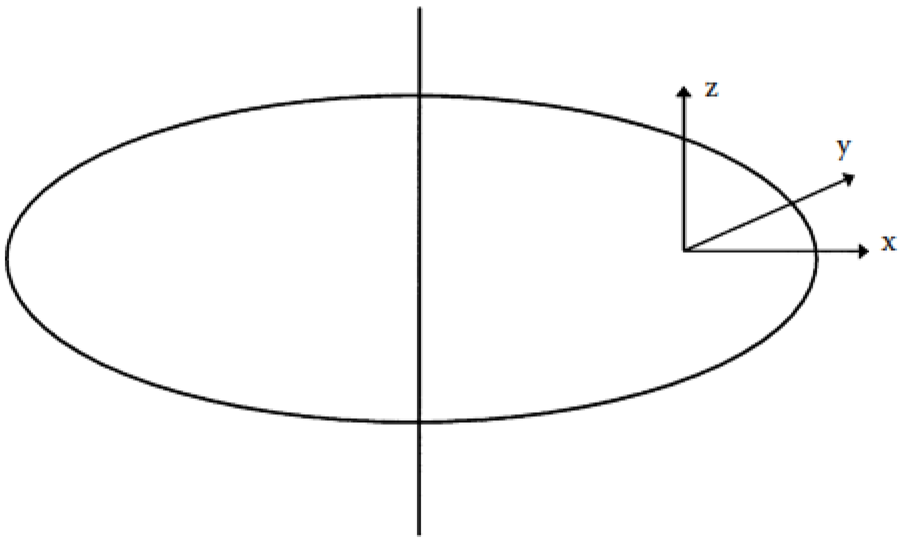


$$\sigma_n = \lim_{\Delta A \to 0} \frac{\Delta P_n}{\Delta A},$$

$$\tau_s = \lim_{\Delta A \to 0} \frac{\Delta P_s}{\Delta A}.$$

Review

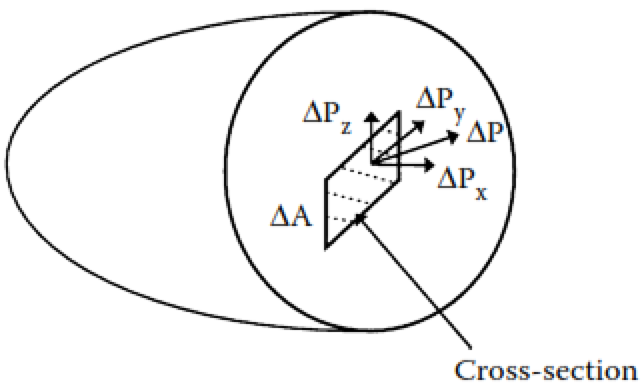
Stress



$$\sigma_x = \lim_{\Delta A \to 0} \frac{\Delta P_x}{\Delta A}$$

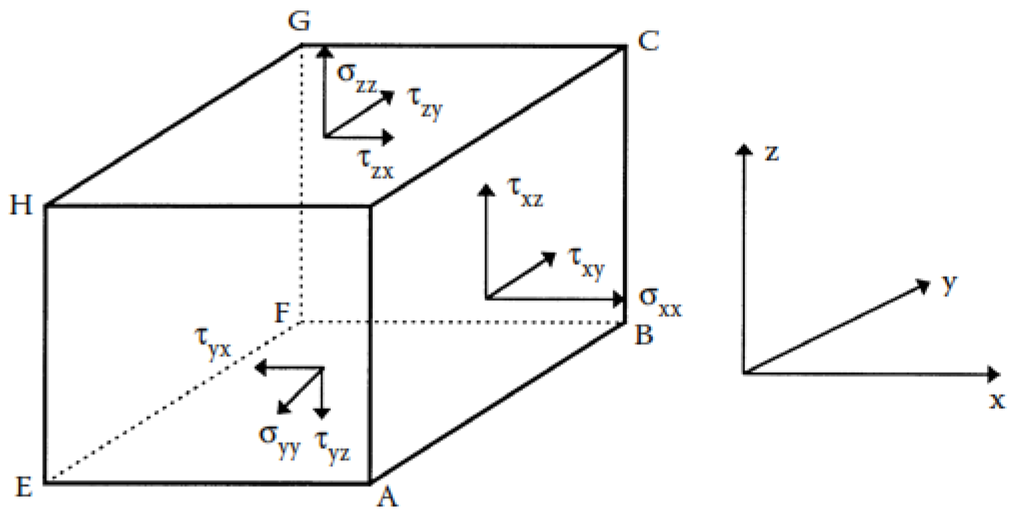
$$\tau_{xy} = \lim_{\Delta A \to 0} \frac{\Delta P_y}{\Delta A} \ ,$$

$$\tau_{xz} = \lim_{\Delta A \to 0} \frac{\Delta P_z}{\Delta A} \ .$$



Review

Stress



$$\tau_{xy} = \tau_{yx},$$

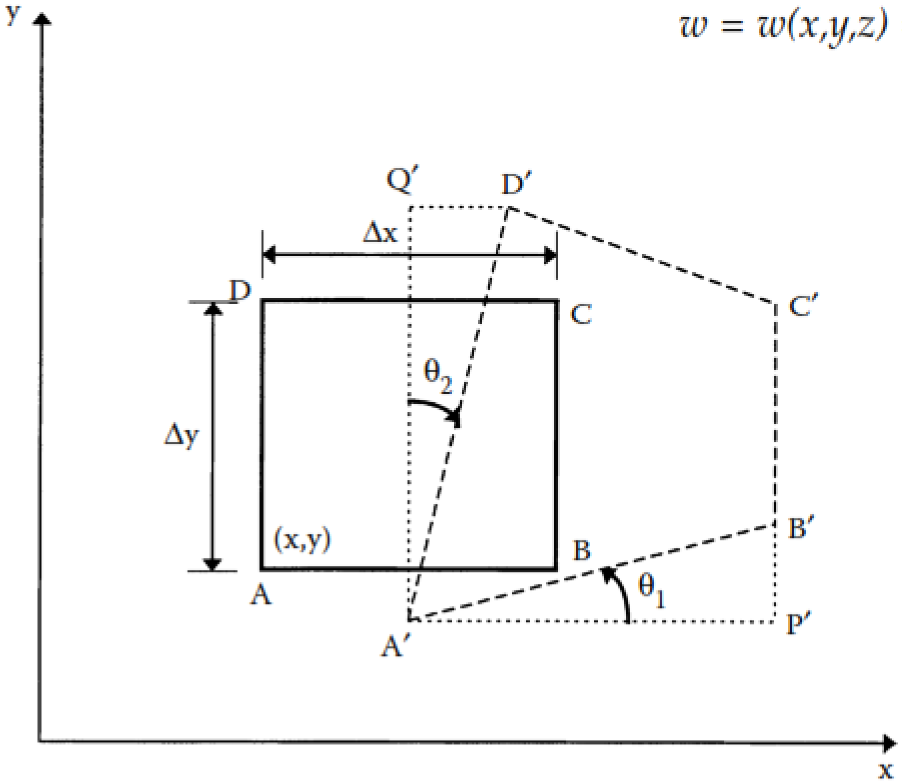
$$\tau_{yz} = \tau_{zy},$$

$$\tau_{zx} = \tau_{xz}.$$

Review

Strain

$u = u(x,y,z)$ = displacement in x-direction at point (x,y,z)
$v = v(x,y,z)$ = displacement in y-direction at point (x,y,z)
$w = w(x,y,z)$ = displacement in z-direction at point (x,y,z)



$$\varepsilon_x = \frac{\partial u}{\partial x},$$

$$\varepsilon_y = \frac{\partial v}{\partial y},$$

$$\varepsilon_z = \frac{\partial w}{\partial z}.$$

Review

Elastic Moduli

3-D stress state – Linear isotropic material, Hooke's Law in a *x*-*y*-*z* orthogonal system:

= [S]

"compliance matrix of an isotropic material"

$$
\begin{bmatrix} \varepsilon_x \\ \varepsilon_y \\ \varepsilon_z \\ \gamma_{yz} \\ \gamma_{zx} \\ \gamma_{xy} \end{bmatrix} = \begin{bmatrix} \frac{1}{E} & -\frac{\nu}{E} & -\frac{\nu}{E} & 0 & 0 & 0 \\ -\frac{\nu}{E} & \frac{1}{E} & -\frac{\nu}{E} & 0 & 0 & 0 \\ -\frac{\nu}{E} & -\frac{\nu}{E} & \frac{1}{E} & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{G} & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{G} & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{G} \end{bmatrix} \begin{bmatrix} \sigma_x \\ \sigma_y \\ \sigma_z \\ \tau_{yz} \\ \tau_{zx} \\ \tau_{xy} \end{bmatrix},
$$

Review

Elastic Moduli

3-D stress state – Linear isotropic material, Hooke's Law in a *x*-*y*-*z* orthogonal system:

$$\begin{bmatrix} \sigma_x \\ \sigma_y \\ \sigma_z \\ \tau_{yz} \\ \tau_{zx} \\ \tau_{xy} \end{bmatrix} = \begin{bmatrix} \frac{E(1-\nu)}{(1-2\nu)(1+\nu)} & \frac{\nu E}{(1-2\nu)(1+\nu)} & \frac{\nu E}{(1-2\nu)(1+\nu)} & 0 & 0 & 0 \\ \frac{\nu E}{(1-2\nu)(1+\nu)} & \frac{E(1-\nu)}{(1-2\nu)(1+\nu)} & \frac{\nu E}{(1-2\nu)(1+\nu)} & 0 & 0 & 0 \\ \frac{\nu E}{(1-2\nu)(1+\nu)} & \frac{\nu E}{(1-2\nu)(1+\nu)} & \frac{E(1-\nu)}{(1-2\nu)(1+\nu)} & 0 & 0 & 0 \\ 0 & 0 & 0 & G & 0 & 0 \\ 0 & 0 & 0 & 0 & G & 0 \\ 0 & 0 & 0 & 0 & 0 & G \end{bmatrix} \begin{bmatrix} \varepsilon_x \\ \varepsilon_y \\ \varepsilon_z \\ \gamma_{yz} \\ \gamma_{zx} \\ \gamma_{xy} \end{bmatrix},$$

= [C]

"stiffness matrix of an isotropic material"

where, $G = \frac{E}{2(1+\nu)}$.

Review

Strain Energy

Energy is defined as the capacity to do work. In solid, deformable, elastic bodies under loads, the work done by external loads is stored as recoverable strain energy. The strain energy stored in the body per unit volume is then defined as

$$W = \frac{1}{2}\left(\sigma_x \varepsilon_x + \sigma_y \varepsilon_y + \sigma_z \varepsilon_z + \tau_{xy} \gamma_{xy} + \tau_{yz} \gamma_{yz} + \tau_{zx} \gamma_{zx}\right).$$

Hooke's Law for different types of Materials

Assuming linear & elastic behaviour for a composite is acceptable, however, assuming it to be isotropic is generally unacceptable!

Anisotropic Material

$$\begin{bmatrix} \sigma_1 \\ \sigma_2 \\ \sigma_3 \\ \tau_{23} \\ \tau_{31} \\ \tau_{12} \end{bmatrix} = \begin{bmatrix} C_{11} & C_{12} & C_{13} & C_{14} & C_{15} & C_{16} \\ C_{21} & C_{22} & C_{23} & C_{24} & C_{25} & C_{26} \\ C_{31} & C_{32} & C_{33} & C_{34} & C_{35} & C_{36} \\ C_{41} & C_{42} & C_{43} & C_{44} & C_{45} & C_{46} \\ C_{51} & C_{52} & C_{53} & C_{54} & C_{55} & C_{56} \\ C_{61} & C_{62} & C_{63} & C_{64} & C_{65} & C_{66} \end{bmatrix} \begin{bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \varepsilon_3 \\ \gamma_{23} \\ \gamma_{31} \\ \gamma_{12} \end{bmatrix},$$

[C] = "stiffness matrix"

Hooke's Law for different types of Materials

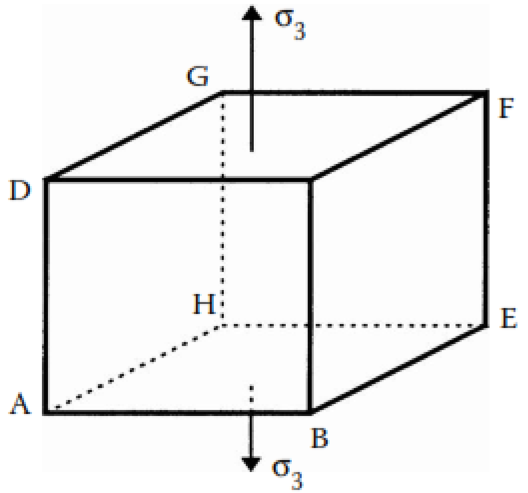
Anisotropic Material

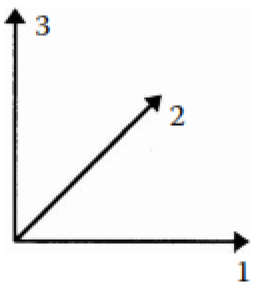
$$\begin{bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \varepsilon_3 \\ \gamma_{23} \\ \gamma_{31} \\ \gamma_{12} \end{bmatrix} = \begin{bmatrix} S_{11} & S_{12} & S_{13} & S_{14} & S_{15} & S_{16} \\ S_{21} & S_{22} & S_{23} & S_{24} & S_{25} & S_{26} \\ S_{31} & S_{32} & S_{33} & S_{34} & S_{35} & S_{36} \\ S_{41} & S_{42} & S_{43} & S_{44} & S_{45} & S_{46} \\ S_{51} & S_{52} & S_{53} & S_{54} & S_{55} & S_{56} \\ S_{61} & S_{62} & S_{63} & S_{64} & S_{65} & S_{66} \end{bmatrix} \begin{bmatrix} \sigma_1 \\ \sigma_2 \\ \sigma_3 \\ \tau_{23} \\ \tau_{31} \\ \tau_{12} \end{bmatrix}.$$

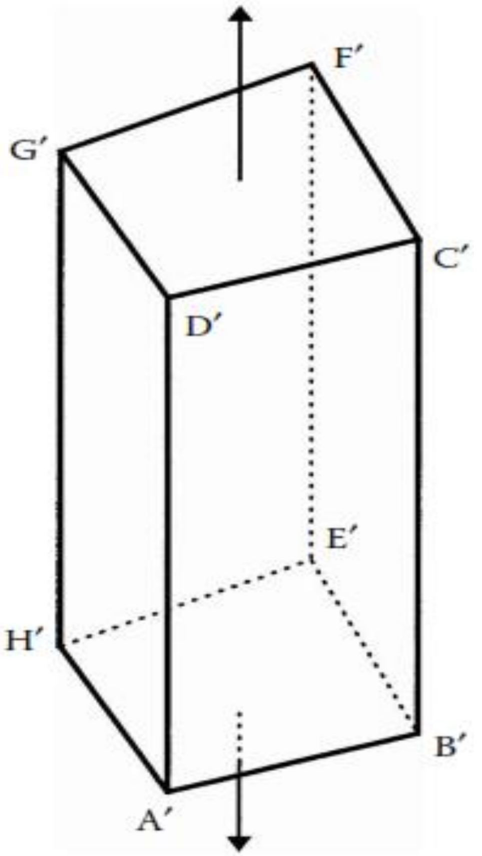
[S] = "compliance matrix"

Hooke’s Law for different types of Materials

Monoclinic Material







Hooke's Law for different types of Materials

Monoclinic Material

If in one plane of material symmetry, for example , direction 3 is normal to the plane of material symmetry, then the stiffness matrix reduces to

$$[C]=\begin{bmatrix} C_{11} & C_{12} & C_{13} & 0 & 0 & C_{16} \\ C_{12} & C_{22} & C_{23} & 0 & 0 & C_{26} \\ C_{13} & C_{23} & C_{33} & 0 & 0 & C_{36} \\ 0 & 0 & 0 & C_{44} & C_{45} & 0 \\ 0 & 0 & 0 & C_{45} & C_{55} & 0 \\ C_{16} & C_{26} & C_{36} & 0 & 0 & C_{66} \end{bmatrix}.$$

Hooke's Law for different types of Materials

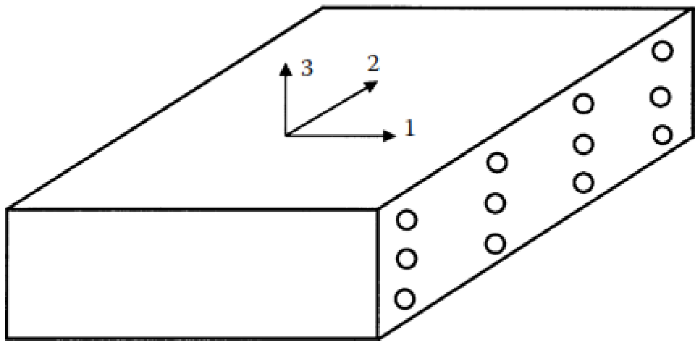
Monoclinic Material

$$[S]=\begin{bmatrix} S_{11} & S_{12} & S_{13} & 0 & 0 & S_{16} \\ S_{12} & S_{22} & S_{23} & 0 & 0 & S_{26} \\ S_{13} & S_{23} & S_{33} & 0 & 0 & S_{36} \\ 0 & 0 & 0 & S_{44} & S_{45} & 0 \\ 0 & 0 & 0 & S_{45} & S_{55} & 0 \\ S_{16} & S_{26} & S_{36} & 0 & 0 & S_{66} \end{bmatrix}.$$

Hooke's Law for different types of Materials

Orthotropic Material

If a material has three mutually perpendicular planes of material symmetry, then the stiffness is given by



$$[C]=\begin{bmatrix} C_{11} & C_{12} & C_{13} & 0 & 0 & 0 \\ C_{12} & C_{22} & C_{23} & 0 & 0 & 0 \\ C_{13} & C_{23} & C_{33} & 0 & 0 & 0 \\ 0 & 0 & 0 & C_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & C_{55} & 0 \\ 0 & 0 & 0 & 0 & 0 & C_{66} \end{bmatrix}.$$

Hooke's Law for different types of Materials

Orthotropic Material

$$[S] = \begin{bmatrix} S_{11} & S_{12} & S_{13} & 0 & 0 & 0 \\ S_{12} & S_{22} & S_{23} & 0 & 0 & 0 \\ S_{13} & S_{23} & S_{33} & 0 & 0 & 0 \\ 0 & 0 & 0 & S_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & S_{55} & 0 \\ 0 & 0 & 0 & 0 & 0 & S_{66} \end{bmatrix}.$$

Hooke's Law for different types of Materials

Orthotropic Material – 12 engineering constants

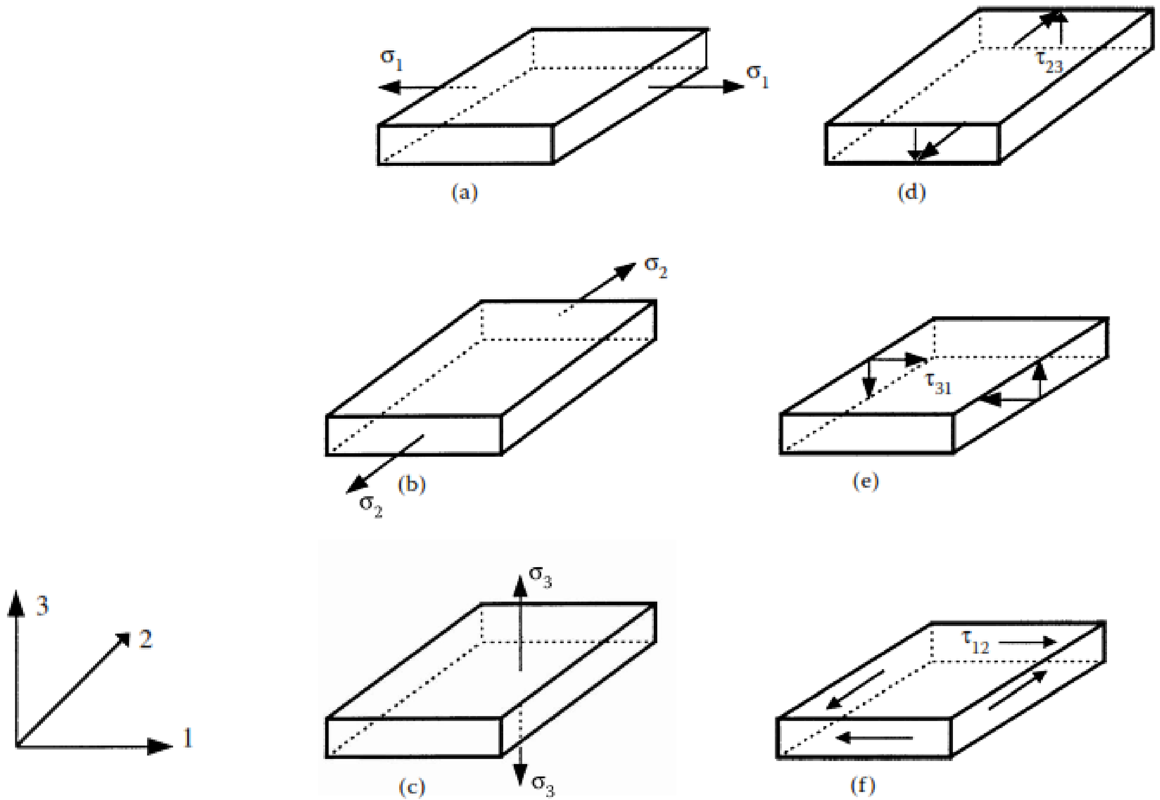


Figure (a),

$\sigma_1 \neq 0,\ \sigma_2 = 0,$

$\sigma_3 = 0,\ \tau_{23} = 0,\ \tau_{31} = 0,\ \tau_{12} = 0.$

Therefore,

$\varepsilon_1 = S_{11}\sigma_1 \qquad \gamma_{23} = 0$

$\varepsilon_2 = S_{12}\sigma_1 \qquad \gamma_{31} = 0$

$\varepsilon_3 = S_{13}\sigma_1 \qquad \gamma_{12} = 0.$

Hooke's Law for different types of Materials

Orthotropic Material – 12 engineering constants

The Young's modulus in direction 1, E_1, is defined as

$$E_1 \equiv \frac{\sigma_1}{\varepsilon_1} = \frac{1}{S_{11}} .$$

The Poisson's ratio, ν_{12}, is defined as

$$\nu_{12} \equiv -\frac{\varepsilon_2}{\varepsilon_1} = -\frac{S_{12}}{S_{11}} .$$

The Poisson's ratio ν_{13} is defined as

$$\nu_{13} \equiv -\frac{\varepsilon_3}{\varepsilon_1} = -\frac{S_{13}}{S_{11}} .$$

Hooke's Law for different types of Materials

Orthotropic Material – 12 engineering constants

Figure (b),

$\sigma_1 = 0,\ \sigma_2 \neq 0,$

$\sigma_3 = 0,\ \tau_{23} = 0,\ \tau_{31} = 0,\ \tau_{12} = 0.$

Therefore,

$$E_2 = \frac{1}{S_{22}}$$

$$\nu_{21} = -\frac{S_{12}}{S_{22}}$$

$$\nu_{23} = -\frac{S_{23}}{S_{22}}\ .$$

Figure (c),

$\sigma_1 = 0,\ \sigma_2 = 0,$

$\sigma_3 \neq 0,\ \tau_{23} = 0,\ \tau_{31} = 0,\ \tau_{12} = 0.$

Therefore,

$$E_3 = \frac{1}{S_{33}}$$

$$\nu_{31} = -\frac{S_{13}}{S_{33}}$$

$$\nu_{32} = -\frac{S_{23}}{S_{33}}\ .$$

Figure (d),

$\sigma_1 = 0,\ \sigma_2 = 0,\ \sigma_3 = 0,\ \tau_{23} \neq 0,\ \tau_{31} = 0,$

$,\ \tau_{12} = 0.$

Therefore,

$\varepsilon_1 = 0 \qquad \gamma_{23} = S_{44}\tau_{23}$

$\varepsilon_2 = 0 \qquad \gamma_{31} = 0$

$\varepsilon_3 = 0 \qquad \gamma_{12} = 0$

The shear modulus in plane 2–3 is defined as

$$G_{23} \equiv \frac{\tau_{23}}{\gamma_{23}} = \frac{1}{S_{44}}\ .$$

Hooke's Law for different types of Materials

Orthotropic Material – 12 engineering constants

Figure (e),

$$\sigma_1 = 0, \sigma_2 = 0,$$

$$\sigma_3 = 0, \tau_{23} = 0, \neq_{31} = 0, \tau_{12} = 0.$$

Therefore,

$$G_{31} = \frac{1}{S_{55}} .$$

Figure (f),

$$\sigma_1 = 0, \sigma_2 = 0,$$

$$\sigma_3 = 0, \tau_{23} = 0, \tau_{31} :\neq 0, \tau_{12} = 0.$$

Therefore,

$$G_{12} = \frac{1}{S_{66}} .$$

Hooke's Law for different types of Materials

Orthotropic Material – 12 engineering constants

12 engineering constants as follow:

Three Young's moduli, E_1, E_2, and E_3, one in each material axis

Six Poisson's ratios, ν_{12}, ν_{13}, ν_{21}, ν_{23}, ν_{31}, and ν_{32}, two for each plane

Three shear moduli, G_{23}, G_{31}, and G_{12}, one for each plane

However, 6 Poisson's ratios are not independent of each other,

$$\frac{\nu_{12}}{E_1} = \frac{\nu_{21}}{E_2}, \qquad \frac{\nu_{13}}{E_1} = \frac{\nu_{31}}{E_3}, \qquad \frac{\nu_{23}}{E_2} = \frac{\nu_{32}}{E_3}.$$

"reciprocal Poison's ratio equations"

Hooke's Law for different types of Materials

Orthotropic Material – 12 engineering constants

Compliance matrix

$$[S]=\begin{bmatrix} \frac{1}{E_1} & -\frac{\nu_{12}}{E_1} & -\frac{\nu_{13}}{E_1} & 0 & 0 & 0 \\ -\frac{\nu_{21}}{E_2} & \frac{1}{E_2} & -\frac{\nu_{23}}{E_2} & 0 & 0 & 0 \\ -\frac{\nu_{31}}{E_3} & -\frac{\nu_{32}}{E_3} & \frac{1}{E_3} & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{G_{23}} & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{G_{31}} & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{G_{12}} \end{bmatrix}.$$

Hooke’s Law for different types of Materials

Orthotropic Material – 12 engineering constants

Stiffness matrix

$$[C]=\begin{bmatrix} \frac{1-\nu_{23}\nu_{32}}{E_2E_3\Delta} & \frac{\nu_{21}+\nu_{23}\nu_{31}}{E_2E_3\Delta} & \frac{\nu_{31}+\nu_{21}\nu_{32}}{E_2E_3\Delta} & 0 & 0 & 0 \\ \frac{\nu_{21}+\nu_{23}\nu_{31}}{E_2E_3\Delta} & \frac{1-\nu_{13}\nu_{31}}{E_1E_3\Delta} & \frac{\nu_{32}+\nu_{12}\nu_{31}}{E_1E_3\Delta} & 0 & 0 & 0 \\ \frac{\nu_{31}+\nu_{21}\nu_{32}}{E_2E_3\Delta} & \frac{\nu_{32}+\nu_{12}\nu_{31}}{E_1E_3\Delta} & \frac{1-\nu_{12}\nu_{21}}{E_1E_2\Delta} & 0 & 0 & 0 \\ 0 & 0 & 0 & G_{23} & 0 & 0 \\ 0 & 0 & 0 & 0 & G_{31} & 0 \\ 0 & 0 & 0 & 0 & 0 & G_{12} \end{bmatrix},$$

Where; $\Delta=\left(1-\nu_{12}\nu_{21}-\nu_{23}\nu_{32}-\nu_{13}\nu_{31}-2\nu_{21}\nu_{32}\nu_{13}\right)/\left(E_1E_2E_3\right).$

Hooke's Law for different types of Materials

Transversely Isotropic Material

Consider a plane of material isotropy in one of the planes of an orthotropic body. If direction 1 is normal to that plane (2-3) of isotropy, the stiffness matrix is given by

$$[C]=\begin{bmatrix} C_{11} & C_{12} & C_{12} & 0 & 0 & 0 \\ C_{12} & C_{22} & C_{23} & 0 & 0 & 0 \\ C_{12} & C_{23} & C_{22} & 0 & 0 & 0 \\ 0 & 0 & 0 & \dfrac{C_{22}-C_{23}}{2} & 0 & 0 \\ 0 & 0 & 0 & 0 & C_{55} & 0 \\ 0 & 0 & 0 & 0 & 0 & C_{55} \end{bmatrix}.$$

Hooke's Law for different types of Materials

Transversely Isotropic Material

Transverse isotropy results in the following relations:

$$C_{22} = C_{33}, C_{12} = C_{13}, C_{55} = C_{66}, C_{44} = \frac{C_{22} - C_{23}}{2} .$$

$$[S] = \begin{bmatrix} S_{11} & S_{12} & S_{12} & 0 & 0 & 0 \\ S_{12} & S_{22} & S_{23} & 0 & 0 & 0 \\ S_{12} & S_{23} & S_{22} & 0 & 0 & 0 \\ 0 & 0 & 0 & 2(S_{22} - S_{23}) & 0 & 0 \\ 0 & 0 & 0 & 0 & S_{55} & 0 \\ 0 & 0 & 0 & 0 & 0 & S_{55} \end{bmatrix} .$$

Hooke's Law for different types of Materials

Isotropic Material

If all planes in an orthotropic body are identical, it is an isotropic material, then the stiffness matrix is given by

$$[C] = \begin{bmatrix} C_{11} & C_{12} & C_{12} & 0 & 0 & 0 \\ C_{12} & C_{11} & C_{12} & 0 & 0 & 0 \\ C_{12} & C_{12} & C_{11} & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{C_{11}-C_{12}}{2} & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{C_{11}-C_{12}}{2} & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{C_{11}-C_{12}}{2} \end{bmatrix}.$$

Hooke's Law for different types of Materials

Isotropic Material

Isotropy results in the following additional relationships:

$$C_{11} = C_{22}, C_{12} = C_{23}, C_{66} = \frac{C_{22} - C_{23}}{2} = \frac{C_{11} - C_{12}}{2} \,.$$

$$C_{11} = \frac{E(1-\nu)}{(1-2\nu)(1+\nu)},$$

$$C_{12} = \frac{\nu E}{(1-2\nu)(1+\nu)}.$$

$$\frac{C_{11} - C_{12}}{2}$$

$$= \frac{1}{2}\left[\frac{E(1-\nu)}{(1-2\nu)(1+\nu)} - \frac{\nu E}{(1-2\nu)(1+\nu)}\right]$$

$$= \frac{E}{2(1+\nu)}$$

$$= G.$$

Hooke's Law for different types of Materials

Isotropic Material

The compliance matrix reduces to

$$[S]=\begin{bmatrix} S_{11} & S_{12} & S_{12} & 0 & 0 & 0 \\ S_{12} & S_{11} & S_{12} & 0 & 0 & 0 \\ S_{12} & S_{12} & S_{11} & 0 & 0 & 0 \\ 0 & 0 & 0 & 2(S_{11}-S_{12}) & 0 & 0 \\ 0 & 0 & 0 & 0 & 2(S_{11}-S_{12}) & 0 \\ 0 & 0 & 0 & 0 & 0 & 2(S_{11}-S_{12}) \end{bmatrix}.$$

Hooke's Law for different types of Materials

Exercise #1

Find the compliance and stiffness matrix for a graphite/epoxy lamina. The material properties are given below:

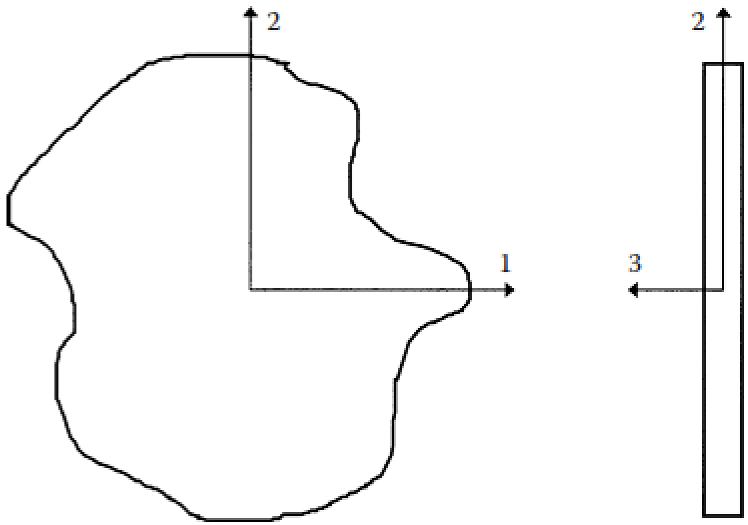
$$E_1 = 181 GPa \ , \ E_2 = 10.3 GPa \ , \ E_3 = 10.3 GPa$$

$$\nu_{12} = 0.28 \ , \ \nu_{23} = 0.60 \ , \ \nu_{13} = 0.27$$

$$G_{12} = 7.17 GPa \ , \ G_{23} = 3.0 GPa \ , \ G_{31} = 7.00 GPa \ .$$

Hooke's Law for a 2D Unidirectional Lamina

Plane Stress Assumption



If a plate is thin and there are no out-of-plane loads. It can be considered to be under plane stress

A lamina is thin, one can assume that it is under plane stress

Therefore

$$\sigma_3 = 0,\ \tau_{31} = 0,\ \tau_{23} = 0.$$

Hooke's Law for a 2D Unidirectional Lamina

Reduction of Hooke's Law in 3D to 2D

Considering, orthotropic plane stress:

$$\begin{bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \gamma_{12} \end{bmatrix} = \begin{bmatrix} S_{11} & S_{12} & 0 \\ S_{12} & S_{22} & 0 \\ 0 & 0 & S_{66} \end{bmatrix} \begin{bmatrix} \sigma_1 \\ \sigma_2 \\ \tau_{12} \end{bmatrix},$$

Inverting the above equation, stress strain relationship as

$$\begin{bmatrix} \sigma_1 \\ \sigma_2 \\ \tau_{12} \end{bmatrix} = \begin{bmatrix} Q_{11} & Q_{12} & 0 \\ Q_{12} & Q_{22} & 0 \\ 0 & 0 & Q_{66} \end{bmatrix} \begin{bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \gamma_{12} \end{bmatrix},$$

= [Q] ; "reduced stiffness coefficients"

$$Q_{11} = \frac{S_{22}}{S_{11}S_{22} - S_{12}^2}, \qquad Q_{12} = -\frac{S_{12}}{S_{11}S_{22} - S_{12}^2}, \qquad Q_{22} = \frac{S_{11}}{S_{11}S_{22} - S_{12}^2}, \qquad Q_{66} = \frac{1}{S_{66}}.$$

Hooke's Law for a 2D Unidirectional Lamina

Relationship of [C] and [S] to Engineering Elastic Constants

For an unidirectional lamina, the engineering elastic constants are:

E_1 = longitudinal Young's modulus (in direction 1)

E_2 = transverse Young's modulus (in direction 2)

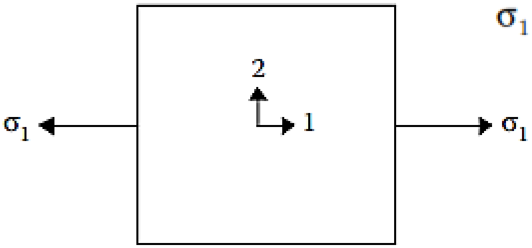
ν_{12} = major Poisson's ratio, where the general Poisson's ratio, ν_{ij} is defined as the ratio of the negative of the normal strain in direction j to the normal strain in direction i, when the only normal load is applied in direction i

G_{12} = in-plane shear modulus (in plane 1–2)

Hooke's Law for a 2D Unidirectional Lamina

Relationship of [C] and [S] to Engineering Elastic Constants

Apply a pure tensile load in direction 1



$$\sigma_1 \neq 0,\ \sigma_2 = 0,\ \tau_{12} = 0.$$

$$\varepsilon_1 = S_{11}\sigma_1,$$

$$\varepsilon_2 = S_{12}\sigma_1,$$

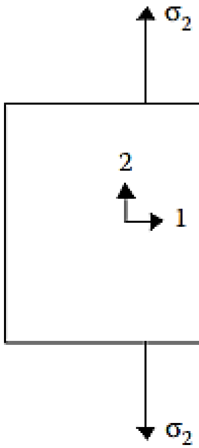
$$\gamma_{12} = 0.$$

If the only nonzero stress is σ_1

$$E_1 \equiv \frac{\sigma_1}{\varepsilon_1} = \frac{1}{S_{11}},$$

$$\nu_{12} \equiv -\frac{\varepsilon_2}{\varepsilon_1} = -\frac{S_{12}}{S_{11}}.$$

Apply a pure tensile load in direction 2



$$\sigma_1 = 0,\ \sigma_2 \neq 0,\ \tau_{12} = 0.$$

$$\varepsilon_1 = S_{12}\sigma_2,$$

$$\varepsilon_2 = S_{22}\sigma_2,$$

$$\gamma_{12} = 0.$$

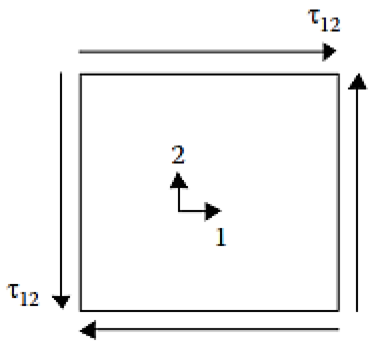
Reciprocal relationship:

$$\frac{\nu_{12}}{E_1} = \frac{\nu_{21}}{E_2}.$$

Hooke's Law for a 2D Unidirectional Lamina

Relationship of [C] and [S] to Engineering Elastic Constants

Apply a pure shear stress in plane 1-2



$$\sigma_1 = 0,\ \sigma_2 = 0 \text{ and } \tau_{12} \neq 0.$$

$$\varepsilon_1 = 0,$$

$$\varepsilon_2 = 0,$$

$$\gamma_{12} = S_{66}\tau_{12}.$$

If the only nonzero stress is τ_{12}

$$G_{12} \equiv \frac{\tau_{12}}{\gamma_{12}} = \frac{1}{S_{66}}.$$

Therefore, proved that:

$$S_{11} = \frac{1}{E_1}, \qquad S_{12} = -\frac{\nu_{12}}{E_1}, \qquad S_{22} = \frac{1}{E_2}, \qquad S_{66} = \frac{1}{G_{12}}.$$

Hooke's Law for a 2D Unidirectional Lamina

Relationship of [C] and [S] to Engineering Elastic Constants

[Q] relationship to the engineering constants

$$Q_{11} = \frac{E_1}{1 - \nu_{21}\nu_{12}}, \qquad Q_{12} = \frac{\nu_{12}E_2}{1 - \nu_{21}\nu_{12}}, \qquad Q_{22} = \frac{E_2}{1 - \nu_{21}\nu_{12}}, \text{ and } \qquad Q_{66} = G_{12}.$$

Summary:

The unidirectional lamina is a *specially orthotropic* lamina because normal stresses applied in the 1–2 direction do not result in any shearing strains in the 1–2 plane because $Q_{16} = Q_{26} = 0 = S_{16} = S_{26}$. Also, the shearing stresses applied in the 1–2 plane do not result in any normal strains in the 1 and 2 directions because $Q_{16} = Q_{26} = 0 = S_{16} = S_{26}$.

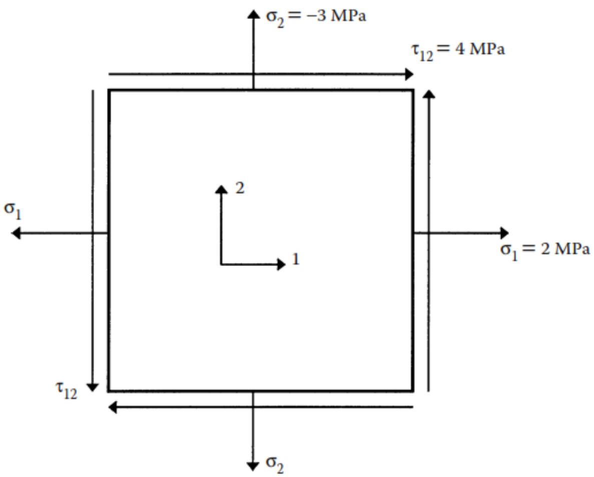
A woven composite with its weaves perpendicular to each other and short fiber composites with fibers arranged perpendicularly to each other or aligned in one direction also are *specially orthotropic*.

Hooke’s Law for a 2D Unidirectional Lamina

Exercise #2

For a graphite/epoxy UD lamina, find the following:

1. Compliance matrix
2. Minor Poisson’s ratio
3. Reduced stiffness matrix
4. Strains in the 1-2 coordinate system if the applied stresses are:



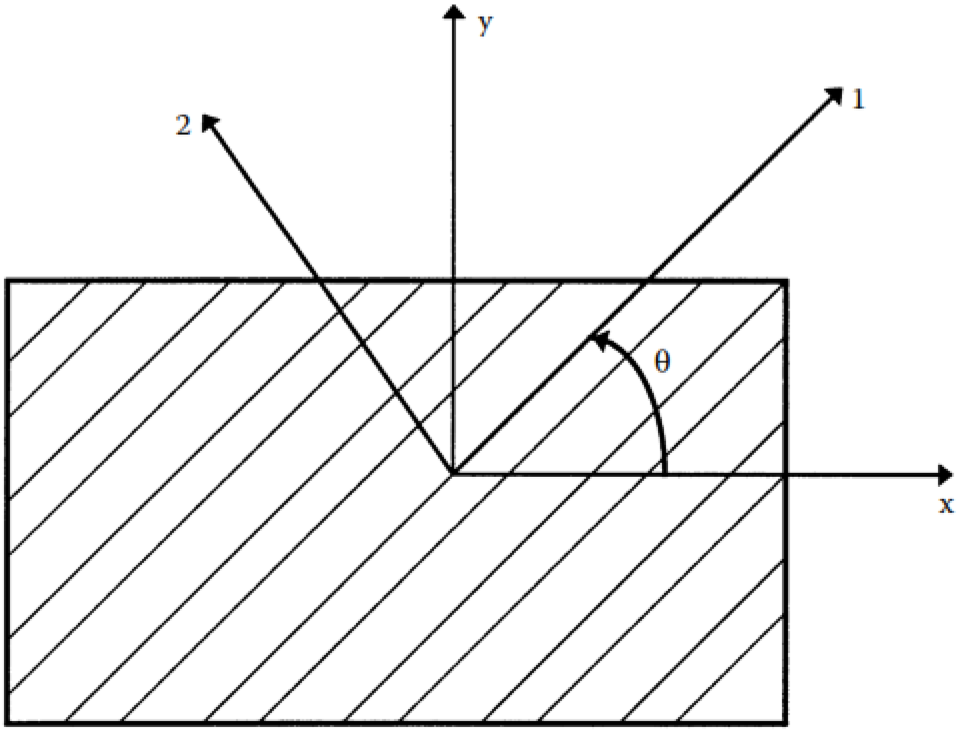
$$\sigma_1 = 2MPa,\ \sigma_2 = -3MPa,\ \tau_{12} = 4MPa.$$

roperties of the material as below:

$$E_1 = 181\ GPa,\ E_2 = 10.3\ GPa,\ \nu_{12} = 0.28,\ G_{12} = 7.17\ GPa.$$

Hooke's Law for a 2D Angle Lamina

Stress-strain Relationship for Angle Lamina



$$\begin{bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{bmatrix} = [T]^{-1} \begin{bmatrix} \sigma_1 \\ \sigma_2 \\ \tau_{12} \end{bmatrix},$$

= $[T]$; "transformation matrix"

where

$$[T]^{-1} = \begin{bmatrix} c^2 & s^2 & -2sc \\ s^2 & c^2 & 2sc \\ sc & -sc & c^2 - s^2 \end{bmatrix},$$

$$[T] = \begin{bmatrix} c^2 & s^2 & 2sc \\ s^2 & c^2 & -2sc \\ -sc & sc & c^2 - s^2 \end{bmatrix},$$

$$c = \mathrm{Cos}(\theta), \qquad s = \mathrm{Sin}(\theta).$$

Hooke's Law for a 2D Angle Lamina

Stress-strain Relationship for Angle Lamina

$$\begin{bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{bmatrix} = \begin{bmatrix} \bar{Q}_{11} & \bar{Q}_{12} & \bar{Q}_{16} \\ \bar{Q}_{12} & \bar{Q}_{22} & \bar{Q}_{26} \\ \bar{Q}_{16} & \bar{Q}_{26} & \bar{Q}_{66} \end{bmatrix} \begin{bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{bmatrix},$$

where

$$\bar{Q}_{11} = Q_{11}c^4 + Q_{22}s^4 + 2(Q_{12} + 2Q_{66})s^2c^2,$$

$$\bar{Q}_{12} = (Q_{11} + Q_{22} - 4Q_{66})s^2c^2 + Q_{12}(c^4 + s^2),$$

$$\bar{Q}_{22} = Q_{11}s^4 + Q_{22}c^4 + 2(Q_{12} + 2Q_{66})s^2c^2,$$

$$\bar{Q}_{16} = (Q_{11} - Q_{12} - 2Q_{66})c^3s - (Q_{22} - Q_{12} - 2Q_{66})s^3c,$$

$$\bar{Q}_{26} = (Q_{11} - Q_{12} - 2Q_{66})cs^3 - (Q_{22} - Q_{12} - 2Q_{66})c^3s,$$

$$\bar{Q}_{66} = (Q_{11} + Q_{22} - 2Q_{12} - 2Q_{66})s^2c^2 + Q_{66}(s^4 + c^4).$$

= $[\bar{Q}]$; "transformed reduced stiffness"

Hooke's Law for a 2D Angle Lamina

Stress-strain Relationship for Angle Lamina

$$\begin{bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{bmatrix} = \begin{bmatrix} \bar{S}_{11} & \bar{S}_{12} & \bar{S}_{16} \\ \bar{S}_{12} & \bar{S}_{22} & \bar{S}_{26} \\ \bar{S}_{16} & \bar{S}_{26} & \bar{S}_{66} \end{bmatrix} \begin{bmatrix} \sigma_x \\ \sigma_y \\ \sigma_{xy} \end{bmatrix},$$

where

$$\bar{S}_{11} = S_{11}c^4 + (2S_{12} + S_{66})s^2c^2 + S_{22}s^4,$$

$$\bar{S}_{12} = S_{12}(s^4 + c^4) + (S_{11} + S_{22} - S_{66})s^2c^2,$$

$$\bar{S}_{22} = S_{11}s^4 + (2S_{12} + S_{66})s^2c^2 + S_{22}c^4,$$

$$\bar{S}_{16} = (2S_{11} - 2S_{12} - S_{66})sc^3 - (2S_{22} - 2S_{12} - S_{66})s^3c,$$

$$\bar{S}_{26} = (2S_{11} - 2S_{12} - S_{66})s^3c - (2S_{22} - 2S_{12} - S_{66})sc^3,$$

$$\bar{S}_{66} = 2(2S_{11} + 2S_{22} - 4S_{12} - S_{66})s^2c^2 + S_{66}(s^4 + c^4).$$

= [S] ; "transformed compliance stiffness"

Hooke's Law for a 2D Angel Lamina

Exercise #3

Find the following for a 60° angle lamina of graphite/epoxy.:

1. Transformed compliance matrix
2. Transformed reduced stiffness matrix

If the applied stress is $\sigma_x = 2$ MPa, $\sigma_y = -3$ MPa, and $\tau_{xy} = 4$ MPa, also find

3. Global strains
4. Local strains
5. Local stresses
6. Principal stresses
7. Maximum shear stress
8. Principal strains
9. Maximum shear strains

Use the properties of the material shown in the next slide:

Property	Symbol	Units	Glass/ epoxy	Boron/ epoxy	Graphite/ epoxy
Fiber volume fraction	V_f		0.45	0.50	0.70
Longitudinal elastic modulus	E_1	GPa	38.6	204	181
Transverse elastic modulus	E_2	GPa	8.27	18.50	10.30
Major Poisson's ratio	ν_{12}		0.26	0.23	0.28
Shear modulus	G_{12}	GPa	4.14	5.59	7.17
Ultimate longitudinal tensile strength	$(\sigma_1^T)_{ult}$	MPa	1062	1260	1500
Ultimate longitudinal compressive strength	$(\sigma_1^C)_{ult}$	MPa	610	2500	1500
Ultimate transverse tensile strength	$(\sigma_2^T)_{ult}$	MPa	31	61	40
Ultimate transverse compressive strength	$(\sigma_2^C)_{ult}$	MPa	118	202	246
Ultimate in-plane shear strength	$(\tau_{12})_{ult}$	MPa	72	67	68
Longitudinal coefficient of thermal expansion	α_1	μm/m/°C	8.6	6.1	0.02
Transverse coefficient of thermal expansion	α_2	μm/m/°C	22.1	30.3	22.5
Longitudinal coefficient of moisture expansion	β_1	m/m/kg/kg	0.00	0.00	0.00
Transverse coefficient of moisture expansion	β_2	m/m/kg/kg	0.60	0.60	0.60

Hooke's Law for a 2D Angel Lamina

Answer #3

1. $$[S] = \begin{bmatrix} 0.8053\times10^{-10} & -0.7878\times10^{-11} & -0.3234\times10^{-10} \\ -0.7878\times10^{-11} & 0.3475\times10^{-10} & -0.4696\times10^{-10} \\ -0.3234\times10^{-10} & -0.4696\times10^{-10} & 0.1141\times10^{-9} \end{bmatrix}^{-1}$$

2. $$[Q] = \begin{bmatrix} 0.2365\times10^{11} & 0.3246\times10^{11} & 0.2005\times10^{11} \\ 0.3246\times10^{11} & 0.1094\times10^{12} & 0.5419\times10^{11} \\ 0.2005\times10^{11} & 0.5419\times10^{11} & 0.3674\times10^{11} \end{bmatrix} Pa.$$

3. $$\begin{bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{bmatrix} = \begin{bmatrix} 0.5534\times10^{-4} \\ -0.3078\times10^{-3} \\ 0.5328\times10^{-3} \end{bmatrix}.$$

4. $$\begin{bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \gamma_{12} \end{bmatrix} = \begin{bmatrix} 0.1367\times10^{-4} \\ -0.2662\times10^{-3} \\ -0.5809\times10^{-3} \end{bmatrix}.$$

Hooke's Law for a 2D Angel Lamina

Answer #3

5. $\begin{bmatrix} \sigma_1 \\ \sigma_2 \\ \tau_{12} \end{bmatrix} := \begin{bmatrix} 0.1714\times10^7 \\ -0.2714\times10^7 \\ -0.4165\times10^7 \end{bmatrix} Pa.$

6. $\sigma_{\max,\min} = \frac{\sigma_x+\sigma_y}{2} \pm \sqrt{\left(\frac{\sigma_x-\sigma_y}{2}\right)^2 + \tau_{xy}^2} = 4.217, -5.217\ MPa.$ $\theta_p = \frac{1}{2}\tan^{-1}\left(\frac{2\tau_{xy}}{\sigma_x-\sigma_y}\right) = 29.00^0$.

7. $\tau_{\max} = \sqrt{\left(\frac{\sigma_x-\sigma_y}{2}\right)^2 + \tau_{xy}^2} = 4.717\ MPa.$ $\theta_s = \frac{1}{2}\tan^{-1}\left(-\frac{\sigma_x-\sigma_y}{2\tau_{xy}}\right) = 16.00^0$

8. $\varepsilon_{\max,\min} = \frac{\varepsilon_x+\varepsilon_y}{2} \pm \sqrt{\left(\frac{\varepsilon_x-\varepsilon_y}{2}\right)^2 + \left(\frac{\gamma_{xy}}{2}\right)^2} = 1.962\times10^{-4}, -4.486\times10^{-4}$. $\theta_p = \frac{1}{2}\tan^{-1}\left(\frac{\gamma_{xy}}{\varepsilon_x-\varepsilon_y}\right) = 27.86^0.$

Hooke's Law for a 2D Angel Lamina

Answer #3

9. $\gamma_{\max} = \sqrt{(\varepsilon_x - \varepsilon_y)^2 + \gamma_{xy}^2} \quad = 6.448 \times 10^{-4}.$

$$\theta_s = \frac{1}{2}\tan^{-1}\left(-\frac{\varepsilon_x - \varepsilon_y}{\gamma_{xy}}\right) \quad = -17.14^0.$$

Engineering Constants of an Angle Lamina

Stress-strain Relationship for Angle Lamina in term of engineering constants;

$$\begin{bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{bmatrix} = \begin{bmatrix} \frac{1}{E_x} & -\frac{\nu_{xy}}{E_x} & -\frac{m_x}{E_1} \\ -\frac{\nu_{xy}}{E_x} & \frac{1}{E_y} & -\frac{m_y}{E_1} \\ -\frac{m_x}{E_1} & -\frac{m_y}{E_1} & \frac{1}{G_{xy}} \end{bmatrix} \begin{bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{bmatrix}.$$

6 constants;

1. $$\frac{1}{E_x} = \bar{S}_{11}$$
$$= S_{11}c^4 + (2S_{12} + S_{66})s^2c^2 + S_{22}s^4.$$
$$= \frac{1}{E_1}c^4 + \left(\frac{1}{G_{12}} - \frac{2\nu_{12}}{E_1}\right)s^2c^2 + \frac{1}{E_2}s^4,$$

2. $$\nu_{xy} = -E_x\bar{S}_{12}$$
$$= -E_x[S_{12}(s^4 + c^4) + (S_{11} + S_{22} - S_{66})s^2c^2]$$
$$= E_x\left[\frac{\nu_{12}}{E_1}(s^4 + c^4) - \left(\frac{1}{E_1} + \frac{1}{E_2} - \frac{1}{G_{12}}\right)s^2c^2\right],$$

Engineering Constants of an Angle Lamina

Stress-strain Relationship for Angle Lamina in term of engineering constants;

6 constants;

3. $$\frac{1}{E_y} = \bar{S}_{22}$$

$$= S_{11}s^4 + (2S_{12} + S_{66})c^2s^2 + S_{22}c^4$$

$$= \frac{1}{E_1}s^4 + \left(-\frac{2\nu_{12}}{E_1} + \frac{1}{G_{12}}\right)c^2s^2 + \frac{1}{E_2}c^4,$$

4. $$\frac{1}{G_{xy}} = \bar{S}_{66}$$

$$= 2(2S_{11} + 2S_{22} - 4S_{12} - S_{66})s^2c^2 + S_{66}(s^4 + c^4)$$

$$= 2\left(\frac{2}{E_1} + \frac{2}{E_2} + \frac{4\nu_{12}}{E_1} - \frac{1}{G_{12}}\right)s^2c^2 + \frac{1}{G_{12}}(s^4 + c^4),$$

Engineering Constants of an Angle Lamina

Stress-strain Relationship for Angle Lamina in term of engineering constants;

6 constants;

5. $$m_x = -\bar{S}_{16}E_1$$

$$= -E_1[(S_{11} - 2S_{12} - S_{66})sc^3 - (2S_{22} - 2S_{12} - S_{66})s^3c]$$

$$= E_1\left[\left(-\frac{2}{E_1} - \frac{2\nu_{12}}{E_1} + \frac{1}{G_{12}}\right)sc^3 + \left(\frac{2}{E_2} + \frac{2\nu_{12}}{E_1} - \frac{1}{G_{12}}\right)s^3c\right],$$

6. $$m_y = -\bar{S}_{26}E_1$$

$$= -E_1[(2S_{11} - 2S_{12} - S_{66})s^3c - (2S_{22} - 2S_{12} - S_{66})sc^3]$$

$$= E_1\left[\left(-\frac{2}{E_1} - \frac{2\nu_{12}}{E_1} + \frac{1}{G_{12}}\right)s^3c + \left(\frac{2}{E_2} + \frac{2\nu_{12}}{E_1} - \frac{1}{G_{12}}\right)sc^3\right].$$

Strength Failure Theories of an Angle Lamina

1. Maximum Stress Failure Theory

Related to MNS theory by Rankine and MSS theory by Tresca

The lamina is considered to be failed if

$$-(\sigma_1^C)_{ult} < \sigma_1 < (\sigma_1^T)_{ult}, \; or$$

$$-(\sigma_2^C)_{ult} < \sigma_2 < (\sigma_2^T)_{ult}, \; or$$

$$-(\tau_{12})_{ult} < \tau_{12} < (\tau_{12})_{ult}$$

Strength Failure Theories of an Angle Lamina

Exercise #4

Find the maximum values of $S > 0$ if a stress of $\sigma_x = 2S$, $\sigma_y = -3S$, and $\tau_{xy} = 4S$
Is applied to the 60° lamina of graphite/epoxy. Use maximum stress failure theory.

$(\sigma_1^T)_{ult} = 1500$ MPa $(\sigma_1^C)_{ult} = 1500$ MPa $(\sigma_2^T)_{ult} = 40$ MPa $(\sigma_2^C)_{ult} = 246$ MPa $(\tau_{12})_{ult} = 68$ MPa

Answer #4

$$\begin{bmatrix} \sigma_1 \\ \sigma_2 \\ \tau_{12} \end{bmatrix} = \begin{bmatrix} 0.2500 & 0.7500 & 0.8660 \\ 0.7500 & 0.2500 & -0.8660 \\ -0.4330 & 0.4330 & -0.5000 \end{bmatrix} \begin{bmatrix} 2S \\ -3S \\ 4S \end{bmatrix}$$

$$= \begin{bmatrix} 0.1714 \times 10^1 \\ -0.2714 \times 10^1 \\ -0.4165 \times 10^1 \end{bmatrix} S.$$

Strength Failure Theories of an Angle Lamina

Answer #4

$$-1500 \times 10^6 < 0.1714 \times 10^1 S < 1500 \times 10^6$$

$$-246 \times 10^6 < -0.2714 \times 10^1 S < 40 \times 10^6$$

$$-68 \times 10^6 < -0.4165 \times 10^1 S < 68 \times 10^6$$

or

$$-875.1 \times 10^6 < S < 875.1 \times 10^6$$

$$-14.73 \times 10^6 < S < 90.64 \times 10^6$$

$$-16.33 \times 10^6 < S < 16.33 \times 10^6.$$

All the inequality conditions (and $S > 0$) are satisfied if $0 < S < 16.33$ MPa. The preceding inequalities also show that the angle lamina will fail in shear. The maximum stress that can be applied before failure is

$$\sigma_x = 32.66\,MPa, \sigma_y = -48.99\,MPa, \tau_{xy} = 65.32\,MPa.$$

Strength Failure Theories of an Angle Lamina

2. Strength Ratio

In a failure theory such as the maximum stress failure theory, it can be determined whether a lamina has failed if any of the inequalities are violated. However, this does not give the information about hoe much the load can be increased if the lamina is safe or how much the load should be decreased if the lamina has failed

The strength ratio is defined as;

$$SR = \frac{\text{Maximum Load Which Can Be Applied}}{\text{Load Applied}}.$$

If $SR > 1$, safe ; $SR < 1$, unsafe; and $SR = 1$ implies the failure load

Strength Failure Theories of an Angle Lamina

Exercise #5

Assume that one is applying a load of $\sigma_x = 2\,MPa, \sigma_y = -3\,MPa, \tau_{xy} = 4\,MPa$ to a 60° angle lamina of graphite/epoxy. Find the strength ratio using the maximum stress failure theory

Answer #5

$$\sigma_x = 2R,\ \sigma_y = -3R,\ \tau_{xy} = 4R\ .$$

$$\sigma_1 = \ 0.1714 \times 10^1\, R$$

$$\sigma_2 = -0.2714 \times 10^1\, R$$

$$\tau_{12} = -0.4165 \times 10^1\, R\ .$$

$$R = 16.33.$$

Strength Failure Theories of an Angle Lamina

3. Failure Envelope

A failure envelope is a three-dimensional plot of the combinations of the NORMAL and SHEAR stresses that can be applied to an angle lamina just before failure. If the applied stress is within the failure envelope, the lamina is safe, otherwise it has failed

Exercise #6

Develop a failure envelope for the 60^o lamina of graphite/epoxy for a constants shear stress of $\tau_{xy} = 24$ MPa.

Strength Failure Theories of an Angle Lamina

Answer #6

$$\sigma_1 = 0.2500\,\sigma_x + 0.7500\,\sigma_y + 20.78\,MPa,$$

$$\sigma_2 = 0.7500\,\sigma_x + 0.2500\,\sigma_y - 20.78\,MPa,$$

$$\tau_{12} = -0.4330\,\sigma_x + 0.4330\,\sigma_y - 12.00\,MPa,$$

$$-1500 < 0.2500\,\sigma_x + 0.7500\,\sigma_y + 20.78 < 1500$$

$$-246 < 0.7500\,\sigma_x + 0.2500\,\sigma_y - 20.78 < 40$$

$$-68 < -0.4330\,\sigma_x + 0.4330\,\sigma_y - 12.00 < 68\ .$$

As another example, for $\sigma_x = 50$ MPa, we have from inequalities,

$$-2044 < \sigma_y < 1956,$$

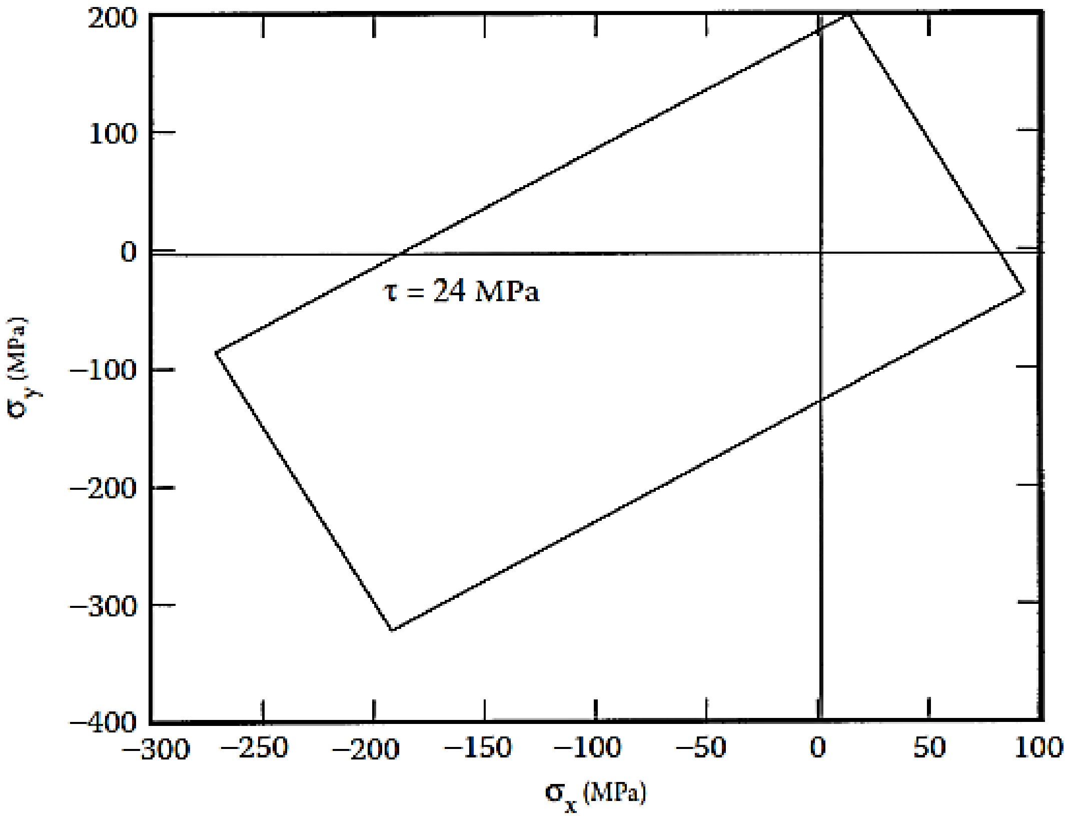
$$-1051 < \sigma_y < 93.12,$$

$$-79.33 < \sigma_y < 234.80.$$

Strength Failure Theories of an Angle Lamina

Answer #6

σ_x (MPa)	σ_y (MPa)
50.0	93.1
50.0	–79.3
–50.0	179
–50.0	–135
25.0	168
25.0	–104
–25.0	160
–25.0	–154



Strength Failure Theories of an Angle Lamina

4. Maximum Strain Failure Theory

Related to MNSt theory by St. Venant and MSS theory by Tresca

The lamina is considered to be failed if

$$-(\varepsilon_1^C)_{ult} < \varepsilon_1 < (\varepsilon_1^T)_{ult}, \text{ or}$$

$$-(\varepsilon_2^C)_{ult} < \varepsilon_2 < (\varepsilon_2^T)_{ult}, \text{ or}$$

$$-(\gamma_{12})_{ult} < \gamma_{12} < (\gamma_{12})_{ult}$$

The ultimate strains can be found directly from the ultimate strength parameters and the elastic moduli, assuming the stress–strain response is linear until failure. The maximum strain failure theory is similar to the maximum stress failure theory in that no interaction occurs between various components of strain. However, the two failure theories give different results because the local strains in a lamina include the Poisson's ratio effect. In fact, if the Poisson's ratio is zero in the unidirectional lamina, the two failure theories will give identical results.

Strength Failure Theories of an Angle Lamina

5. Tsai-Hill Failure Theory

Based on the distortion energy failure theory of Von-Mises distortional energy yield criterion. Distortion energy is a part of the total strain energy in a body. The strain energy in a body consists of two parts; one due to a change in volume and is called the dilation energy and the second is due to change in shape and is called the distortion energy. Hill adopted the Von-Mises distortional energy yield criterion to anisotropic materials. Then, Tsai adapted it to an unidirectional lamina.

The lamina is considered to be failed if

$$(G_2+G_3)\sigma_1^2+(G_1+G_3)\sigma_2^2+(G_1+G_2)\sigma_3^2-2G_3\sigma_1\sigma_2-2G_2\sigma_1\sigma_3$$

$$-2G_1\sigma_2\sigma_3+2G_4\tau_{23}^2+2G_5\tau_{13}^2+2G_6\tau_{12}^2<1$$

Reduce the equation to:

$$\left[\frac{\sigma_1}{(\sigma_1^T)_{ult}}\right]^2-\left[\frac{\sigma_1\sigma_2}{(\sigma_1^T)_{ult}^2}\right]+\left[\frac{\sigma_2}{(\sigma_2^T)_{ult}}\right]^2+\left[\frac{\tau_{12}}{(\tau_{12})_{ult}}\right]^2<1.$$

Strength Failure Theories of an Angle Lamina

6. Tsai-Wu Failure Theory

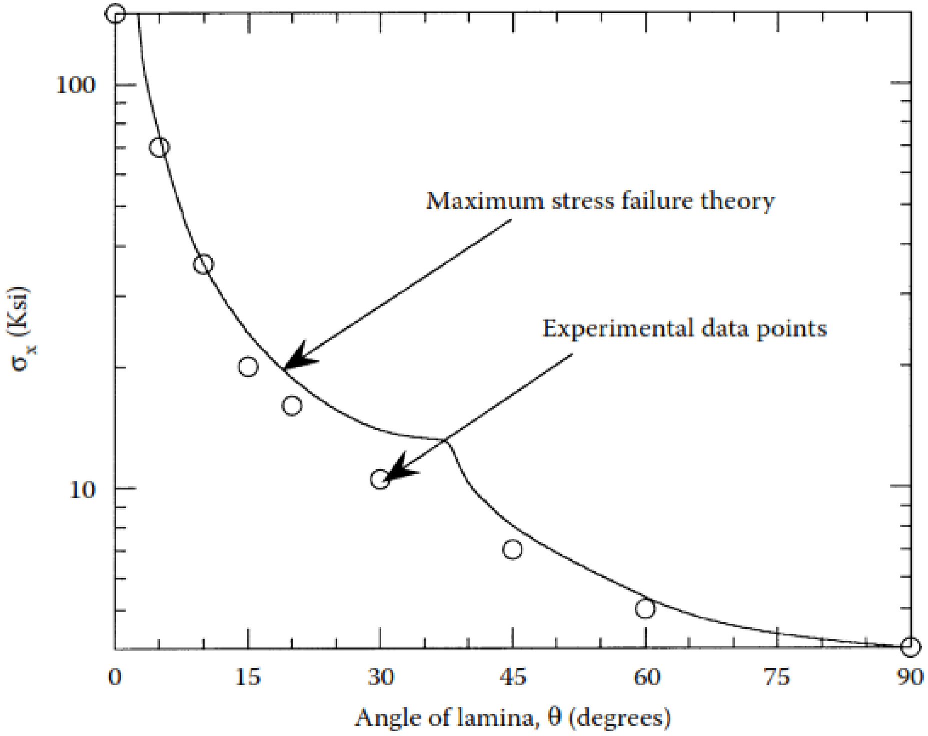
Based on the total strain energy failure theory of Beltrami. Tsai-Wu applied the failure theory to a lamina in plane stress. This failure theory is more general than the Tsai-Hill failure theory because it distinguishes between the compressive and tensile strengths of a lamina.

The lamina is considered to be failed if

$$H_1\sigma_1 + H_2\sigma_2 + H_6\tau_{12} + H_{11}\,\sigma_1^2 + H_{22}\,\sigma_2^2 + H_{66}\,\tau_{12}^2 + 2H_{12}\sigma_1\sigma_2 < 1$$

Strength Failure Theories of an Angle Lamina

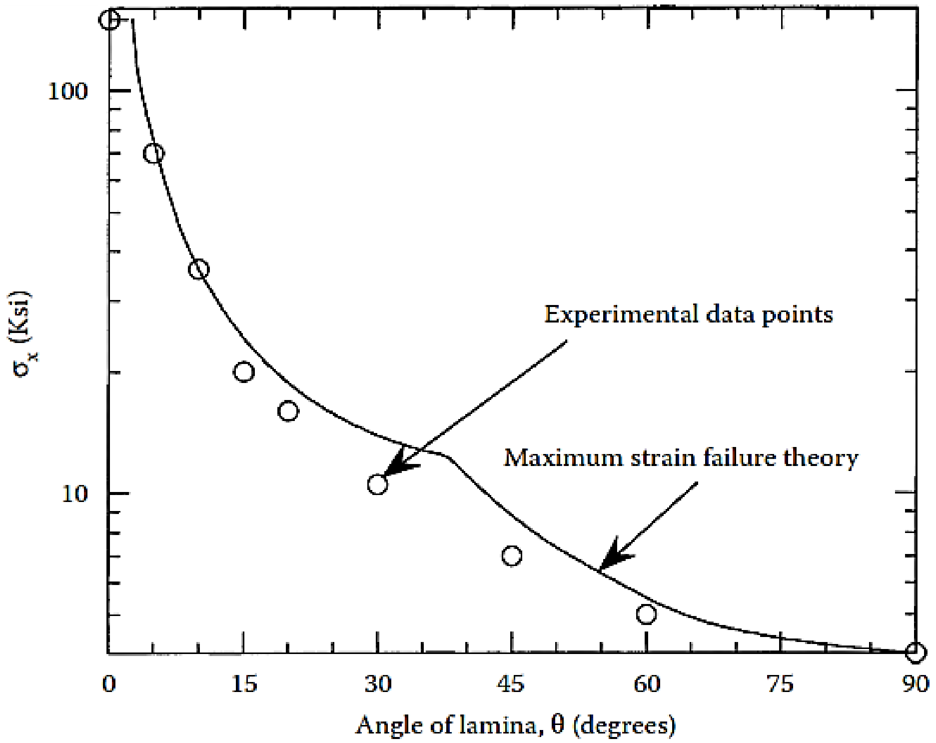
7. Comparison of Experimental Results with Failure Theories



Maximum normal tensile stress in the x-direction as a function of angle of lamina using maximum stress failure theory. (Experimental data reprinted with permission from *Introduction to Composite Materials*, Tsai, S.W. and Hahn, H.T., 1980, CRC Press, Boca Raton, FL, 301.)

Strength Failure Theories of an Angle Lamina

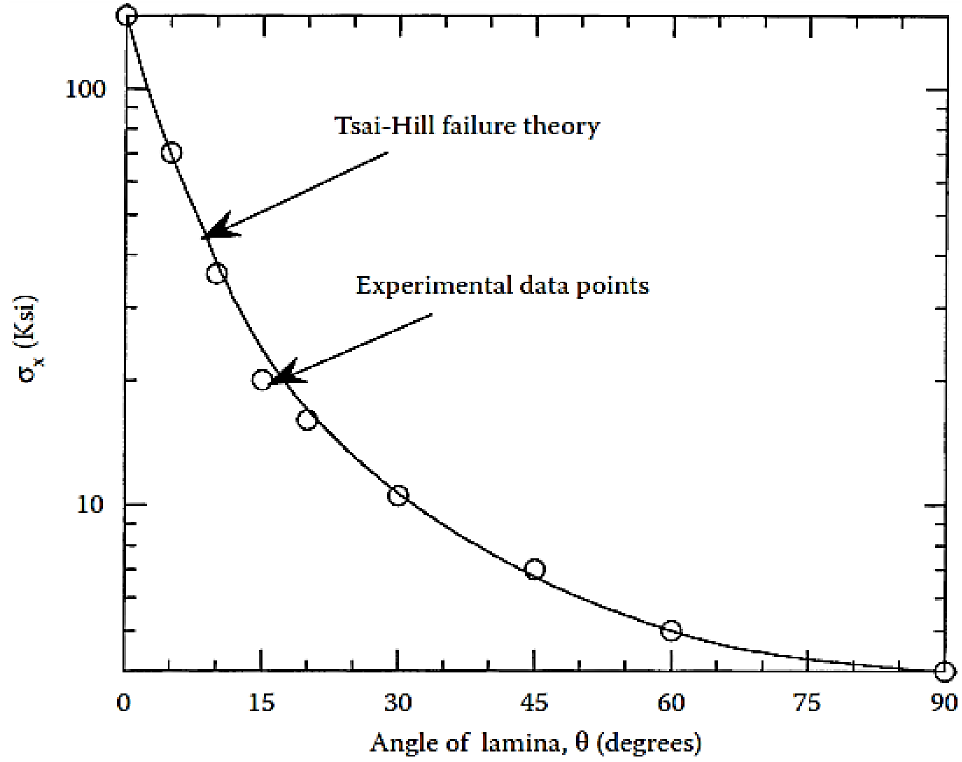
7. Comparison of Experimental Results with Failure Theories



Maximum normal tensile stress in the *x*-direction as a function of angle of lamina using maximum strain failure theory. (Experimental data reprinted with permission from *Introduction to Composite Materials*, Tsai, S.W. and Hahn, H.T., 1980, CRC Press, Boca Raton, FL, 301.)

Strength Failure Theories of an Angle Lamina

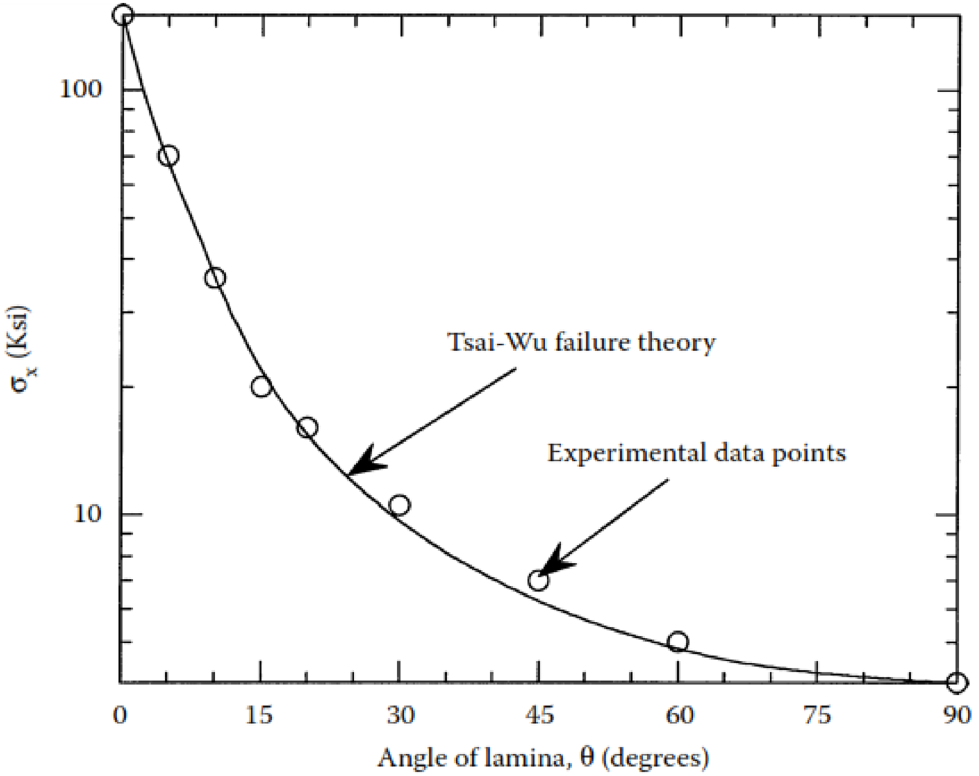
7. Comparison of Experimental Results with Failure Theories



Maximum normal tensile stress in the *x*-direction as a function of angle of lamina using Tsai–Hill failure theory. (Experimental data reprinted with permission from *Introduction to Composite Materials*, Tsai, S.W. and Hahn, H.T., 1980, CRC Press, Boca Raton, FL, 301.)

Strength Failure Theories of an Angle Lamina

7. Comparison of Experimental Results with Failure Theories



Maximum normal tensile stress in the *x*-direction as a function of angle of lamina using Tsai–Wu failure theory. (Experimental data reprinted with permission from *Introduction to Composite Materials*, Tsai, S.W. and Hahn, H.T., 1980, CRC Press, Boca Raton, FL, 301.)

Strength Failure Theories of an Angle Lamina

7. Comparison of Experimental Results with Failure Theories

Observations:

- The difference between the maximum stress and maximum strain failure theories and the experimental results is quite pronounced.
- Tsai–Hill and Tsai–Wu failure theories' results are in good agreement with experimentally obtained results.
- The variation of the strength of the angle lamina as a function of angle is smooth in the Tsai–Hill and Tsai–Wu failure theories, but has cusps in the maximum stress and maximum strain failure theories. The cusps correspond to the change in failure modes in the maximum stress and maximum strain failure theories.

Strength Failure Theories of an Angle Lamina

Exercise #7

Find the maximum values of $S > 0$ if a stress of $\sigma_x = 2S,\ \sigma_y = -3S$, and $\tau_{xy} = 4S$ is applied to the 60º lamina of graphite/epoxy. Use Tsai-Hill failure theory.

$(\sigma_1^T)_{ult} = 1500$ MPa $\quad (\sigma_1^C)_{ult} = 1500$ MPa $\quad (\sigma_2^T)_{ult} = 40$ MPa $\quad (\sigma_2^C)_{ult} = 246$ MPa $\quad (\tau_{12})_{ult} = 68$ MPa

Answer #7

$\sigma_1 = 1.714\ S, \qquad \sigma_2 = -2.714\ S, \qquad \tau_{12} = -4.165\ S.$

$$\left(\frac{1.714S}{1500\times10^6}\right)^2 - \left(\frac{1.714S}{1500\times10^6}\right)\left(\frac{-2.714S}{1500\times10^6}\right) + \left(\frac{-2.714S}{40\times10^6}\right)^2 + \left(\frac{-4.165S}{68\times10^6}\right)^2 < 1$$

$$S < 10.94\ MPa$$

Thank you

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