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THEORY OF STRUCTURES
CHAPTER 2 : DEFLECTION (MOMENT AREA METHOD)
PART 3

by
Saffuan Wan Ahmad
Faculty of Civil Engineering & Earth Resources
saffuan@ump.edu.my

Chapter 2 : Part 3 – Unit Load Method

- Aims
  - Determine the slope and deflection by using Moment Area Method

- Expected Outcomes :
  - Able to analyze determinate beam – deflection and slope by Moment Area Method.

- References

  - Mechanics of Materials, R.C. Hibbeler, 7th Edition, Prentice Hall
  - Structural Analysis, Hibbeler, 7th Edition, Prentice Hall
  - Structural Analysis, SI Edition by Aslam Kassimali,Cengage Learning
  - Structural Analysis, Coates, Coatie and Kong
  - Structural Analysis - A Classical and Matrix Approach, Jack C. McCormac and James K. Nelson, Jr., 4th Edition, John Wiley

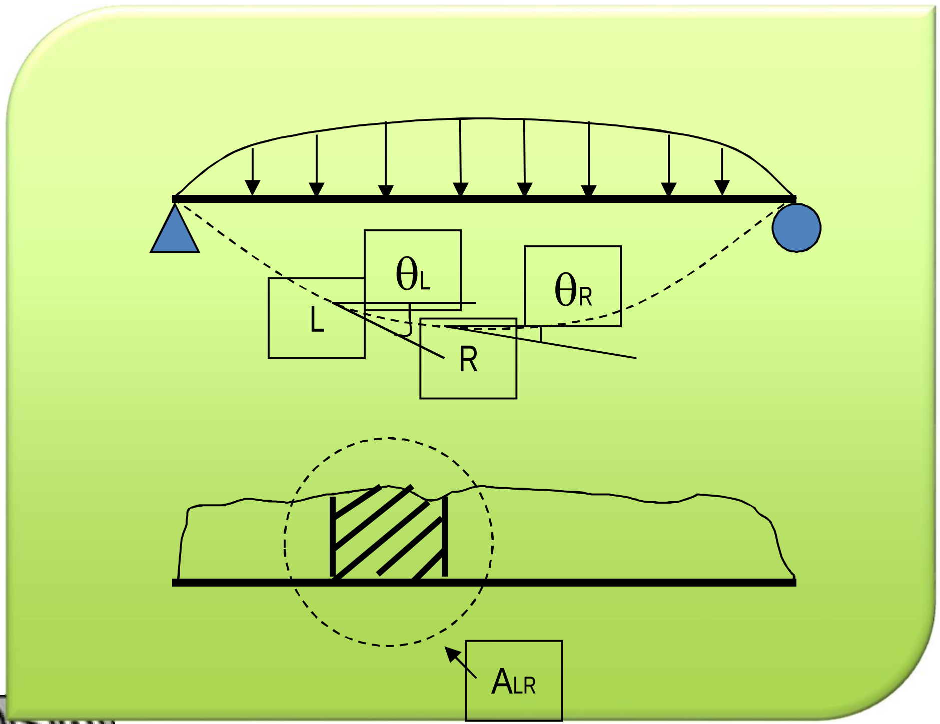
MOMENT AREA METHOD

- Moment area method: based on following theorem where the relationship between bending moment, slope and deflection
- Was developed by Otto Mohr in 1873
- Provide a semi-graphical technique for determining slope and deflection

MOMENT AREA METHOD

Theorem I:

- Any point *L* is located to the left of any other point *R*
- The difference of slope between the points *L* and *R* is equal to the area of (M/EI) diagram between the two points



Where,

θ_L, θ_R =slope at points L and R

A_{LR} = area of M/EI diagram between points L and R

MOMENT AREA METHOD

a)Slope

We know that,

$$\frac{1}{R} = \frac{M}{EI}$$

The slope at any point is tan θ = *dy/dx* where θ is small then tan θ = θ so,

$$\frac{dy}{dx} = \theta \Rightarrow \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{dx}(\theta)$$

$$\frac{d^2y}{dx^2} = \frac{d\theta}{dx}$$

MOMENT AREA METHOD

We also know that the elastic curve equation:

$$EI\frac{d^2y}{dx^2} = M$$

Therefore,

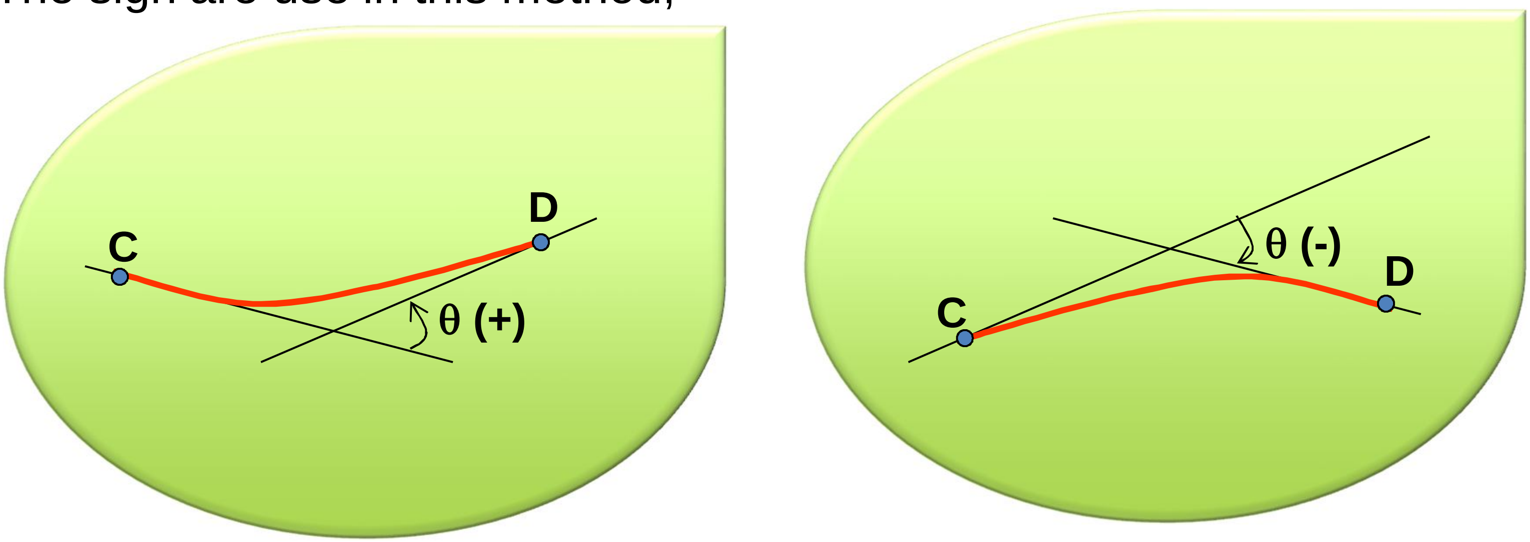
$$EI\frac{d\theta}{dx} = M$$

$$d\theta = \frac{M}{EI}dx$$

$$\therefore \theta_{LR} = \int_{\theta_R}^{\theta_L} d\theta = \frac{1}{EI}\int_{x_R}^{x_L} Mdx$$

MOMENT AREA METHOD

The sign are use in this method,

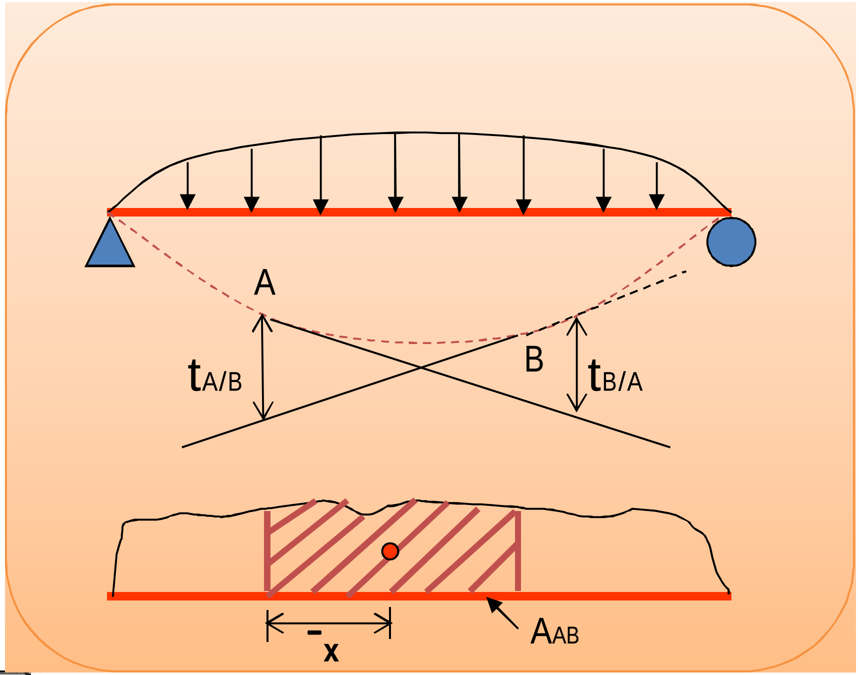


If the direction for 1st tangent line (normally left point) to 2nd tangent line is counter-clockwise, the change of slope become positive (+)

MOMENT AREA METHOD

Theorem II:

- The transverse (vertical deviation) displacement of any point A measured from the tangent to the deflection curve at any other point B is equal to the *'moment'* about A of the area of (M/EI) diagram between A and B ($t_{A/B}$)



Where,

$\bar{x}$ = distance of centroid of A_{AB} from A

A_{AB} = area of (M/EI) diagram between points A and B

MOMENT AREA METHOD

a)Deflection

If $d\theta$ is small and the distance between centroid point to AB is x, therefore:

$$dt = xd\theta = x\frac{M}{EI}dx$$

Integrating the equation from A to B, then we can write as:

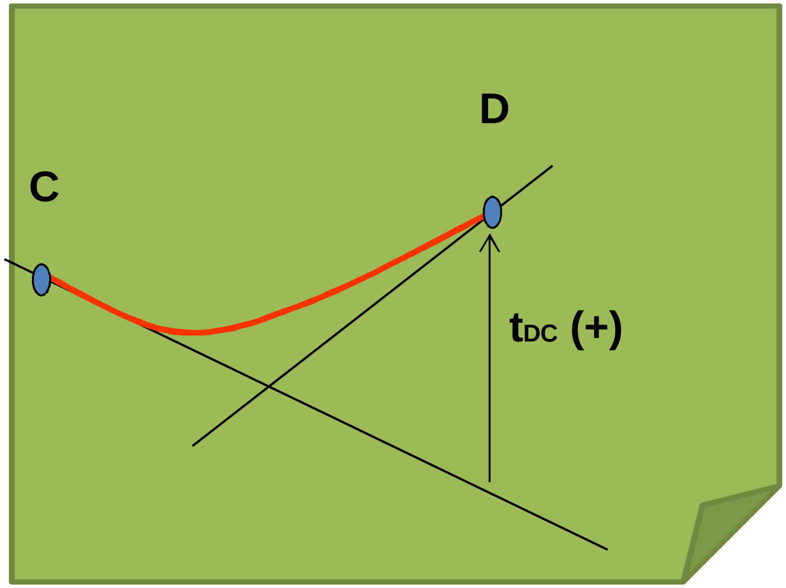
$$t_{AB} = \int_{t_B}^{t_A} dt$$

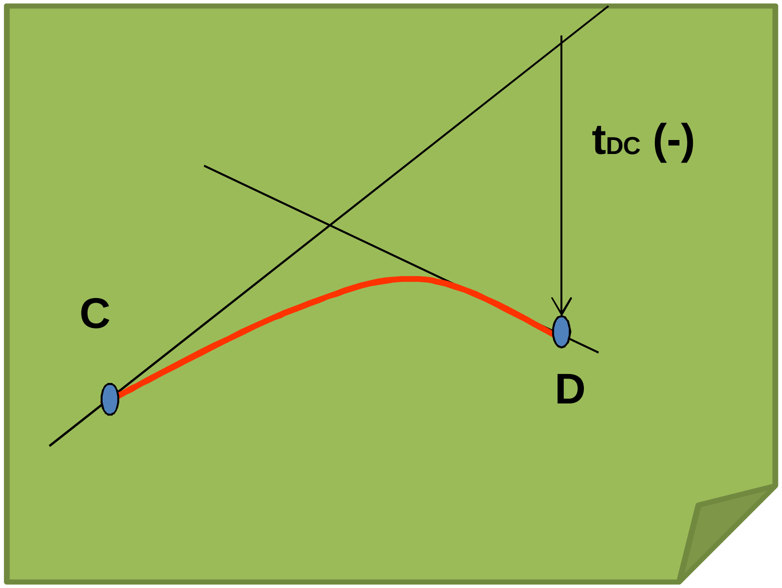
$$= \int_{x_B}^{x_A} x\frac{M}{EI}dx$$

$$= \frac{1}{EI}\int_{x_B}^{x_A} \bar{x}Mdx$$

MOMENT AREA METHOD

The sign are use in this method,





If the point D located above tangent line at C, then vertical distance between tangent line at D and C (tDC) is become positive (+)

MOMENT AREA METHOD

$$Theorem\ I:$$

(Slope changing)

$$\theta_{xy} = \frac{1}{EI}(Area\ M_{x-y})$$

$$Theorem\ II:$$

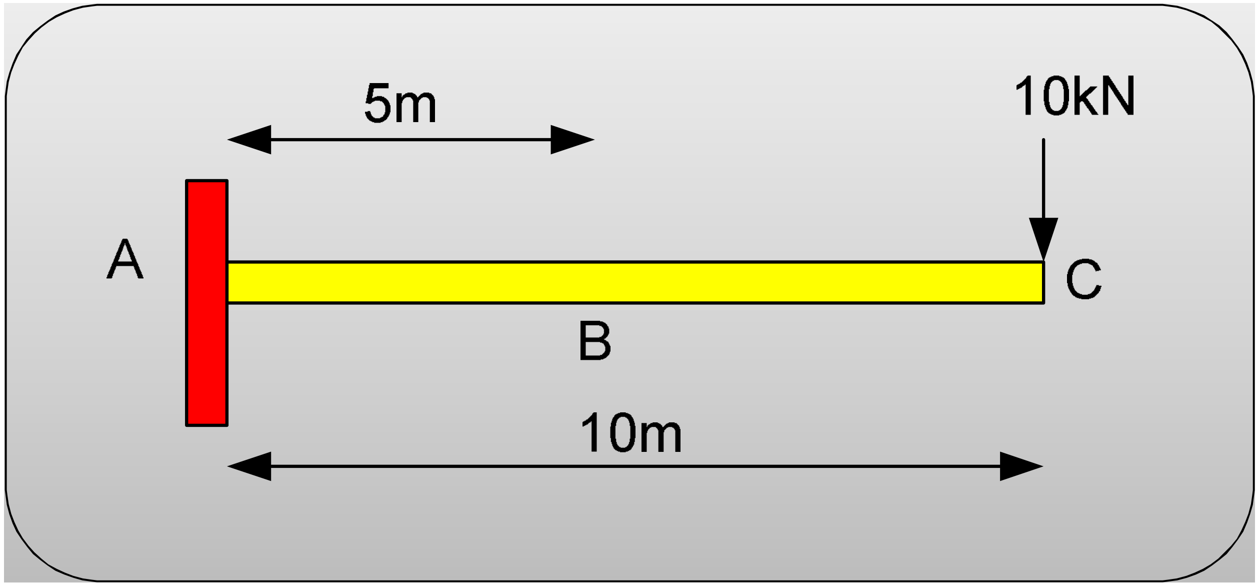
(Deflection changing)

$$t_{xy} = \frac{1}{EI}(Area\ M_{x-y})(\bar{x})$$

$\bar{x}$ is the distance from the vertical axis of the point to the centroid of the moment area

EXAMPLE 1

DETERMINE THE SLOPE AT POINTS B AND C OF THE BEAM SHOWN BELOW. TAKE E = 200 GPa AND
$I = 360 \times 10^6\ mm^4$



$$\theta_B = \theta_{B/A}; \quad \theta_C = \theta_{C/A}$$

- Applying Theorem 1, is equal to the area under the M/EI diagram between points A & B

$$\theta_B = \theta_{B/A} = -\left(\frac{50kNm}{EI}\right)(5m) - \frac{1}{2}\left(\frac{100kNm}{EI} - \frac{50kNm}{EI}\right)(5m)$$

$$= -\frac{375kNm^2}{EI}$$

- Substituting numerical data for E & I

$$-\frac{375kNm^2}{[200(10^6)kN/m^2][360(10^6)(10^{-12})m^4]} = -0.00521rad$$

- The –ve sign indicates that the angle is measured clockwise from A, Fig 8.15(c)
- In a similar manner, the area under the M/EI diagram between points A & C equals ($\theta_{C/A}$)

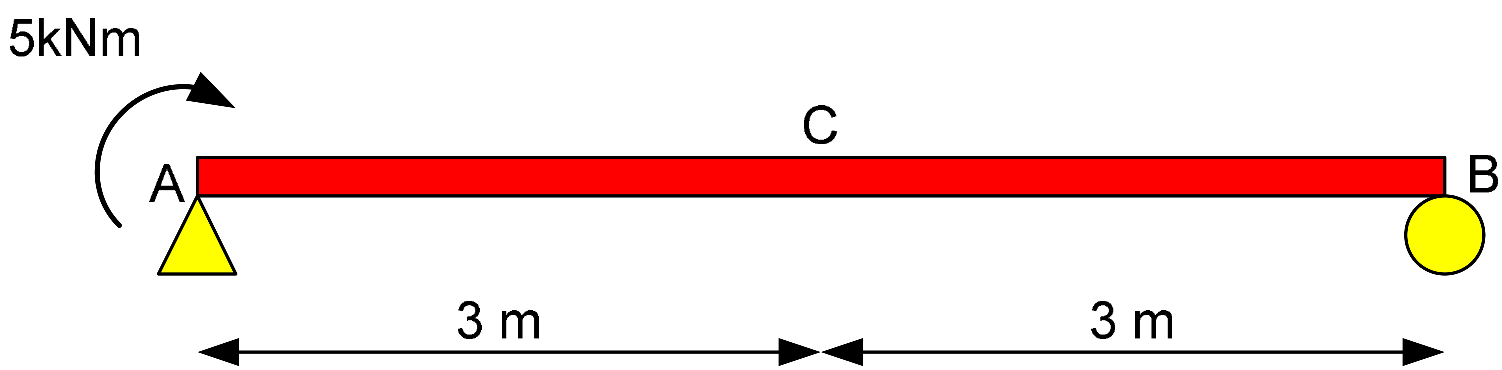
$$\theta_C = \theta_{C/A} = \frac{1}{2}\left(-\frac{100\,kNm}{EI}\right)(10\,m) = -\frac{500\,kNm^2}{EI}$$

Substituti ng numerical values of EI, we have :

$$\frac{-500\,kNm^2}{[200(10^6)kN/m^2][360(10^6)(10^{-12})m^4]} = -0.00694\,rad$$

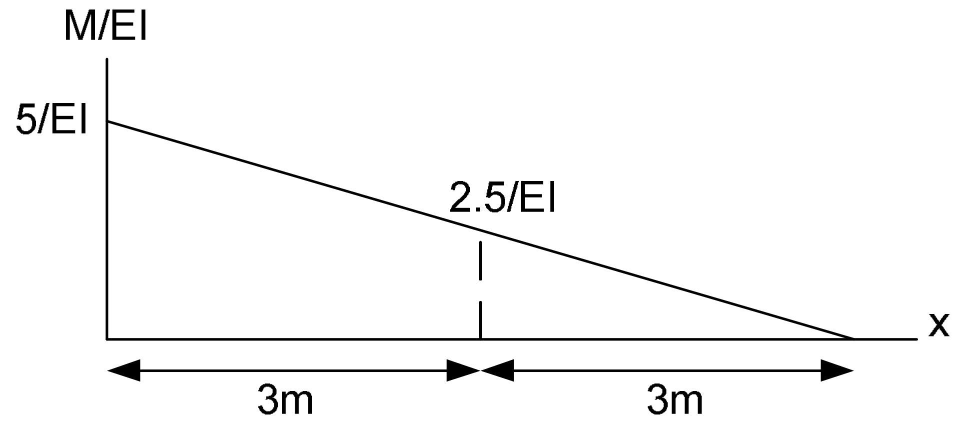
EXAMPLE 2

Determine the deflection at C of the beam shown as below. Take E = 200GPa and I=360E6 mm4.

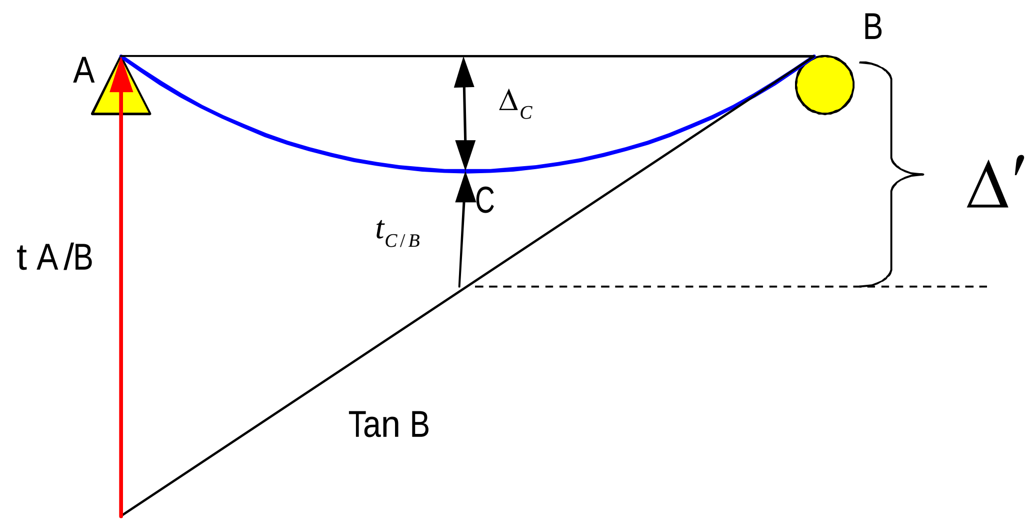


Solution

M/EI Diagram



ELASTIC CURVE



From the elastic curve

$$\frac{\Delta'}{3} = \frac{t_{A/B}}{6}$$

OR

$$\Delta' = \frac{t_{A/B}}{2}$$

$$\Delta_C = \frac{t_{A/B}}{2} - t_{C/B} \qquad (1)$$

Moment area theorem. ---- **apply theorem 2**

$t_{A/B}$ is the moment of the M/EI diagram between A and B about point A

$$t_{A/B} = \left[\frac{1}{3}(6m)\right]\left[\frac{1}{2}(6m)\left(\frac{5kN.m}{EI}\right)\right]$$

$$t_{A/B} = \frac{30kN.m^3}{EI}$$

$t_{C/D}$ is the moment of the M/EI diagram between C and B about the point C

$$t_{C/B} = \left[\frac{1}{3}(3m)\right]\left[\frac{1}{2}(3m)\left(\frac{2.5kN.m}{EI}\right)\right]$$

$$t_{C/B} = \frac{3.75kN.m^3}{EI}$$

SUBSTITUTING THESE RESULT INTO EQU 1

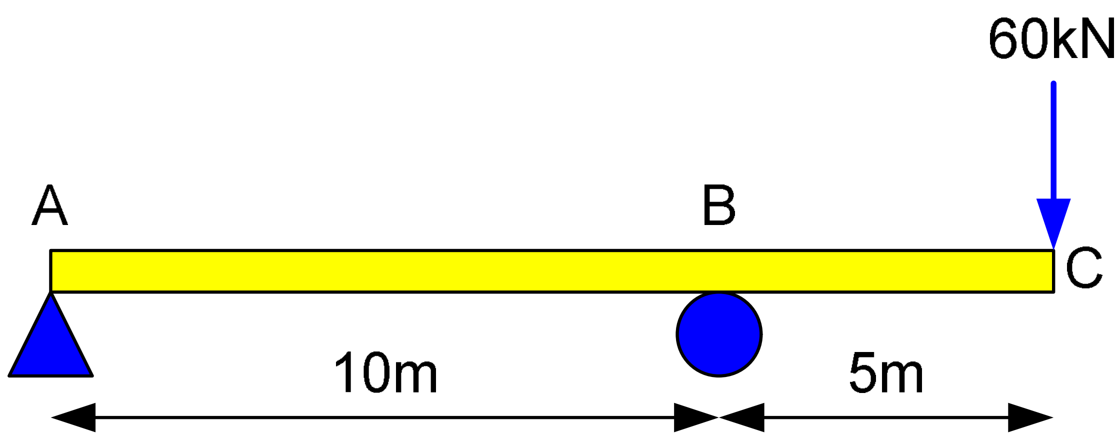
$$\Delta_C = \frac{1}{2}\left(\frac{30kN.m^3}{EI}\right) - \frac{3.75kN.m^3}{EI}$$

$$\Delta_C = \frac{11.25kN.m^3}{EI}$$

$$\Delta_C = 0.0141m = 14.1mm$$

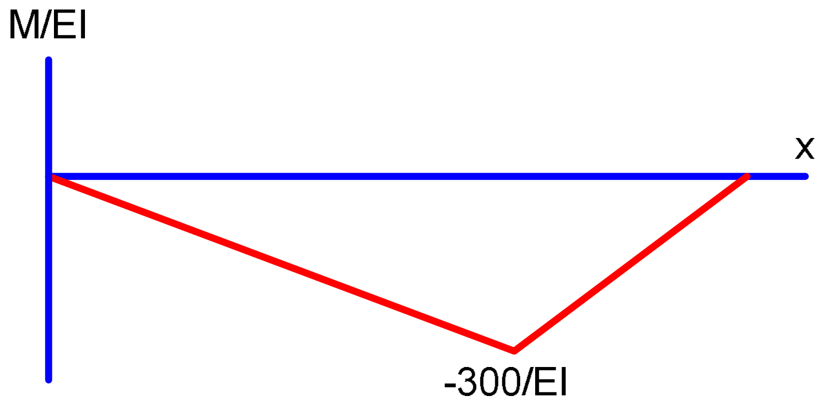
EXAMPLE 3

USE THE MOMENT AREA THEOREM TO DETERMINE THE DEFLECTION AT C. EI IS CONSTANT.

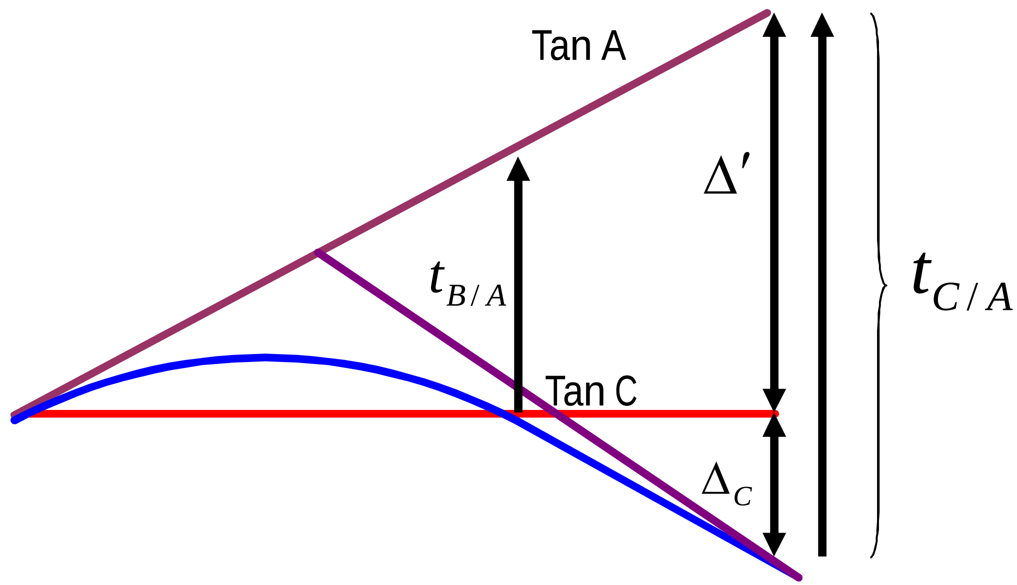


solution

M/EI Diagram



Elastic Curve



Deflection at C

From elastic curve,

$$\frac{\Delta'}{15} = \frac{t_{B/A}}{10}$$

$$\Delta' = \frac{15}{10} t_{B/A}$$

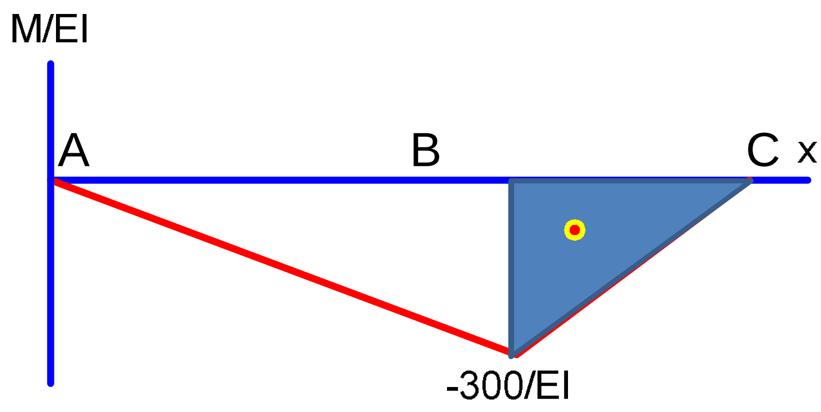
$$\Delta_C = t_{C/A} - \Delta'$$

$$\Delta_C = t_{C/A} - \frac{15}{10} t_{B/A}$$

$t_{C/A}$ **is the moment of the M/EI diagram between C and A about C**

$$t_{C/A} = \left[\frac{1}{2}\left(\frac{-300}{EI}\right)(5)(3.33)\right] + \left[\frac{1}{2}\left(\frac{-300}{EI}\right)(10)(8.33)\right]$$

$$t_{C/A} = \frac{-15000kN.m^3}{EI}$$



tB/A **is the moment of the M/EI diagram between B and A about B**

$$t_{B/A} = \left[\frac{1}{2} \left(\frac{-300}{EI} \right) (10)(3.33) \right]$$

$$t_{C/A} = \frac{-5000 kN.m^3}{EI}$$

SUBSTITUTING THESE RESULT INTO Δ_C EQUATION

$$\Delta_C = \frac{-15000}{EI} - \frac{15}{10}\left(\frac{-5000}{EI}\right)$$

$$\Delta_C = \frac{7500kN.m^3}{EI}$$

Bending Moment Diagrams by Parts

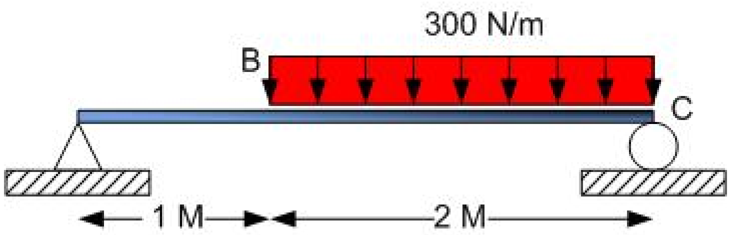
- Application of the moment-area theorems is practically only if the area under the bending moment diagrams and its first moment can be calculated without difficulty.
- The key to simplifying the computation is to divide the BMD into simple geometric shape (rectangles, triangles and parabolas) that have known areas and centroidal coordinates.
- Sometimes the conventional BMD lends itself to such division, but often it is preferable to draw the BMD by parts, with each part of the diagrams representing the effect of one load.

Construction BMD by parts for simply supported beam

- Calculate the support reactions
- introduce a **FIX SUPPORT** as a convinient location.

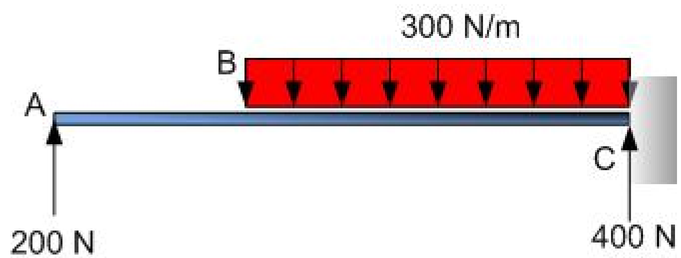
A simply support by the original beam is usually a good choice, but sometimes another point is more convenient. The beam is now cantilevered from this support.

- Draw a BMD for each loading (including the support reactions of the original beam. If all the diagrams can be fitted on a single plot, do so. Draw the positive moment above the x-axis and negative moment below the x-axis



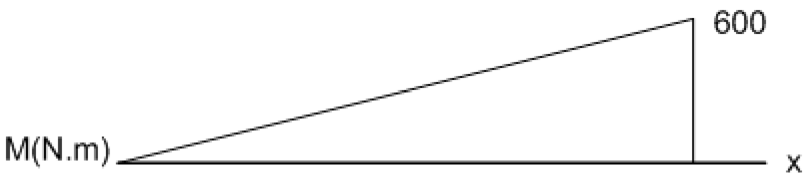
Calculate the reactions at support

(a)

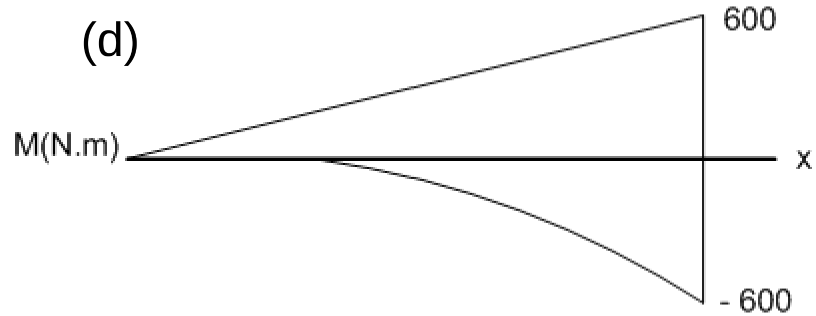


Introduce fixed support at point C.. So Now draw BMD due to support 200 N and UDL in parts

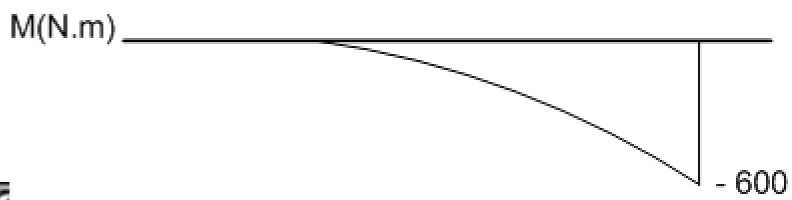
(b)



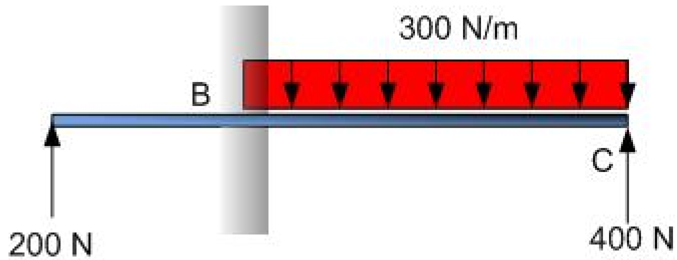
(d)



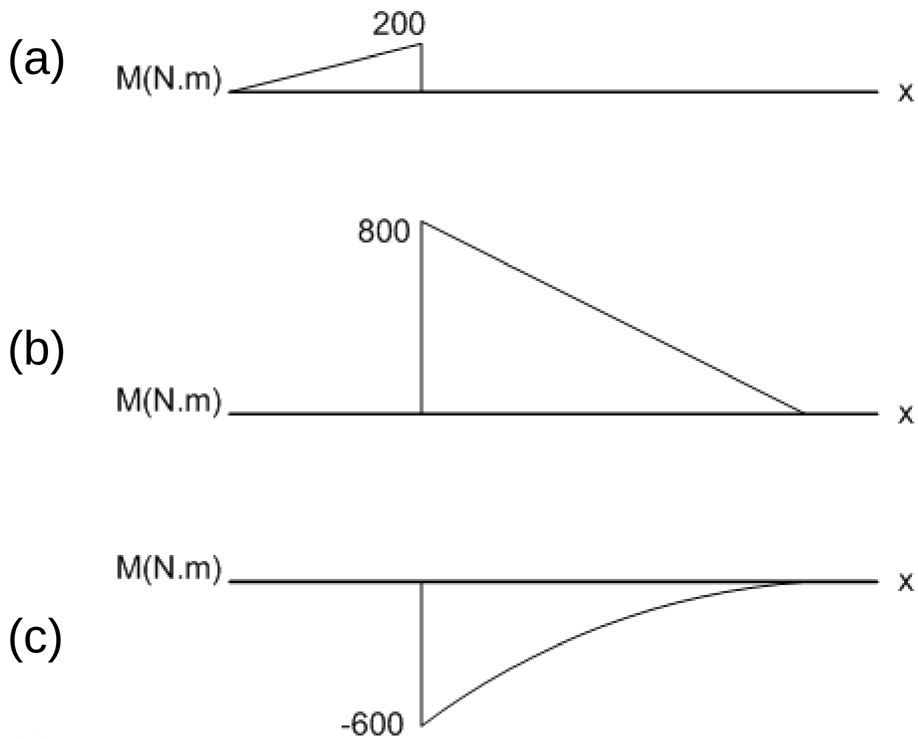
(c)

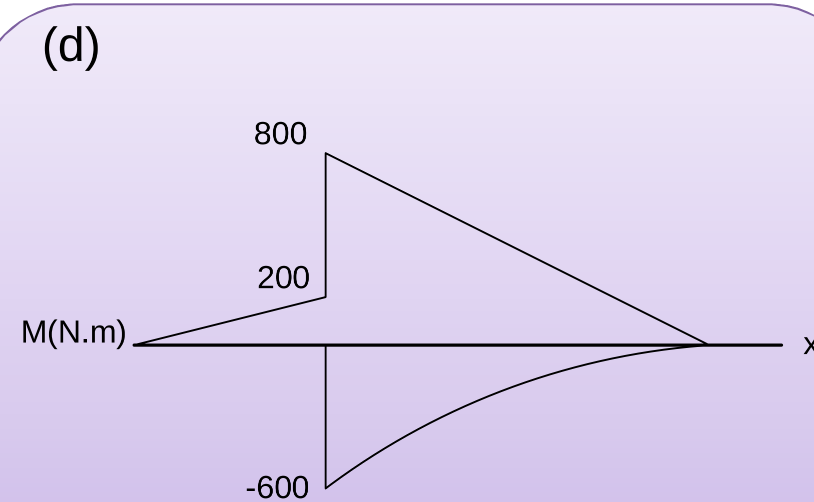


(a) Equivalent beam with fixed support at C, (b) BMD due to support reaction at A, (c) BMD due to UDL,(d) combination BMD by parts



If ...
We introduce fixed support at B, than the BMD contains three parts

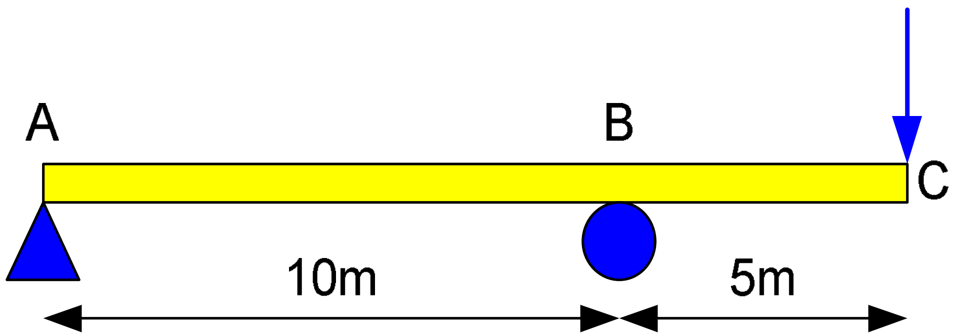




(a) BMD due to reaction at A, (b) BMD due to support reation at C, (c) BMD due to UDL,(d) combination BMD by parts

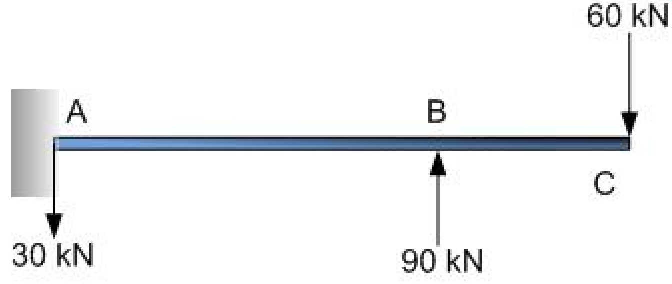
EXAMPLE 4

USE THE MOMENT AREA THEOREM TO DETERMINE THE DEFLECTION AT C.EI IS CONSTANT.

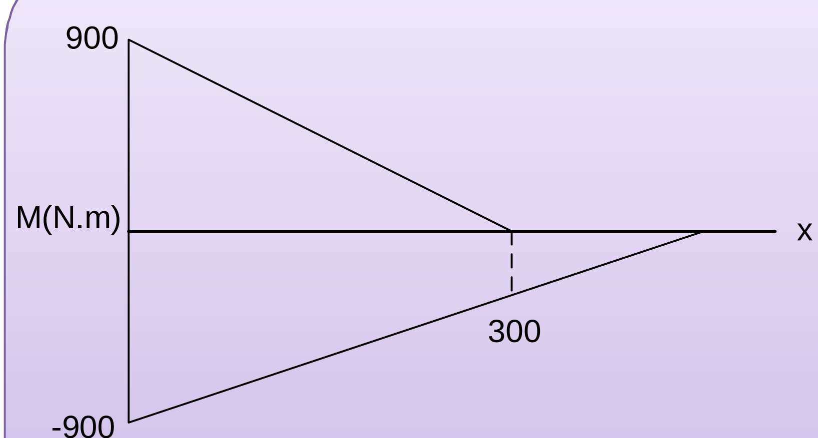


This example same to the previous example. Try solve this problem by developed BMD by part. Introduce point A as fixed support

Calculate the reactions at support

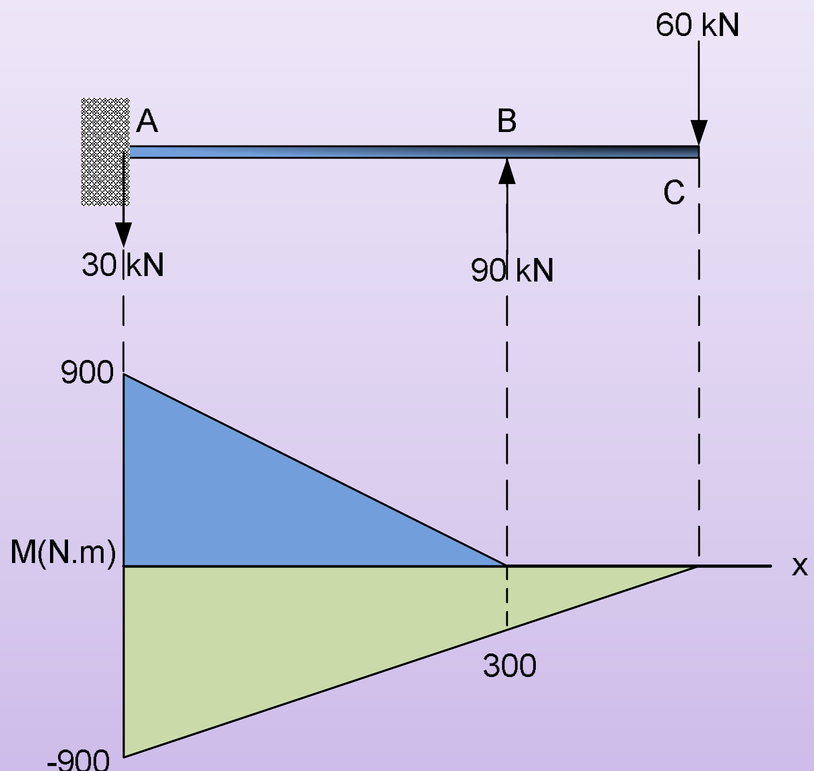


Introduce fixed support at point A.. So Now draw BMD due to support 90 kN and point load of 60 kN in parts



From the elastic curve… refer to the previous example

$$\Delta_C = t_{C/A} - \frac{15}{10} t_{B/A}$$



$$t_{C/A} = \frac{1}{2}(900)(10)(11.67)$$

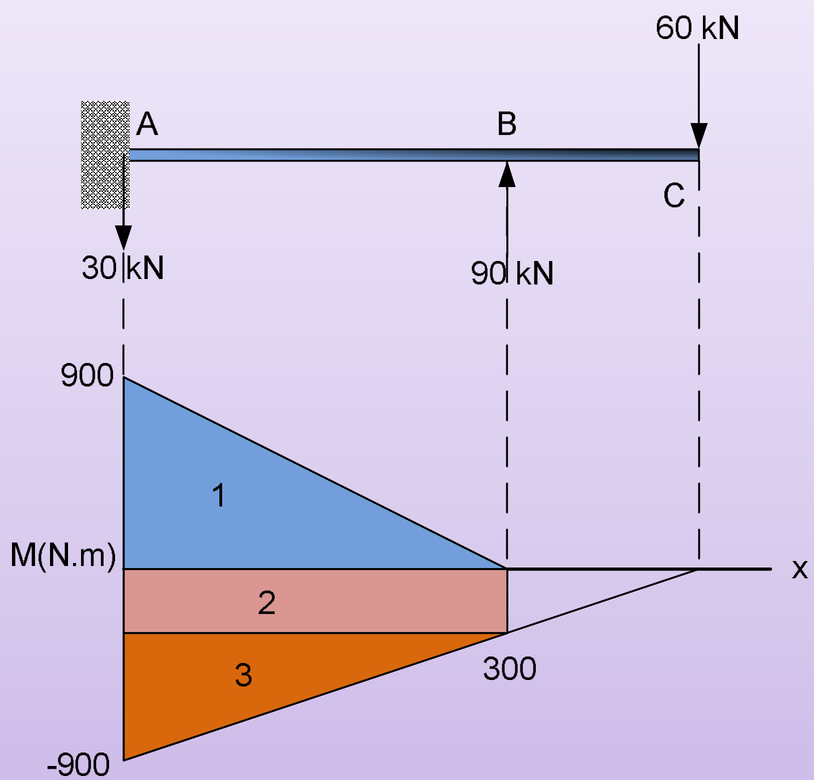
$$+\frac{1}{2}(-900)(15)(\frac{2}{3})(15)$$

$$t_{C/A} = 52500 - 67500$$

$$= -15000$$

From the elastic curve… refer to the previous example

$$\Delta_C = t_{C/A} - \frac{15}{10} t_{B/A}$$



$$t_{B/A} = \frac{1}{2}(900)(10)(\frac{2}{3})(10)$$

$$+(-300)(10)(\frac{1}{2})(10)$$

$$+\frac{1}{2}(-600)(10)(\frac{2}{3})(10)$$

$$t_{B/A} = 30000 - 15000 - 20000$$

$$= -5000$$

SUBSTITUTING THESE RESULT INTO Δ_C EQUATION

$$\Delta_C = \frac{-15000}{EI} - \frac{15}{10}\left(\frac{-5000}{EI}\right)$$

$$\Delta_C = \frac{7500 kN.m^3}{EI}$$

THANKS

Author Information

Mohd Arif Bin Sulaiman
Mohd Faizal Bin Md. Jaafar
Mohammad Amirulkhairi Bin Zubir
Rokiah Binti Othman
Norhaiza Binti Ghazali
Shariza Binti Mat Aris