For updated version, please click on http://ocw.ump.edu.my

THEORY OF STRUCTURES
CHAPTER 2 : DEFLECTION (UNIT LOAD METHOD)
PART 2

by
Saffuan Wan Ahmad
Faculty of Civil Engineering & Earth Resources
saffuan@ump.edu.my

Chapter 2 : Part 2 – Unit Load Method

- Aims
  - Determine the slope and deflection by using Unit Load Method

- Expected Outcomes :
  - Able to analyze determinate beam – deflection and slope by Unit Load Method

- References
  - Mechanics of Materials, R.C. Hibbeler, 7th Edition, Prentice Hall
  - Structural Analysis, Hibbeler, 7th Edition, Prentice Hall
  - Structural Analysis, SI Edition by Aslam Kassimali,Cengage Learning
  - Structural Analysis, Coates, Coatie and Kong
  - Structural Analysis - A Classical and Matrix Approach, Jack C. McCormac and James K. Nelson, Jr., 4th Edition, John Wiley

Principle of Virtual Work

- The internal work in transversely loaded beams is taken equal to the strain energy due to bending moment
- The virtual force F_i in the i^{th} mass element in $\Delta = F^*_i e_i$ may be taken equal to the bending moment m_{ij} in the i^{th} mass element due to a unit load at coordinate j

Principle of Virtual Work (Displacement)

- Sometimes referred as the Unit-Load Method
- Generally provides of obtaining the displacement and slope at a specific point on structure i.e. beam, frame or truss
- In general, the principle states that:

Work of Internal Loads

$$\sum P\Delta = \sum u\delta$$

Work of External Loads

Principle of Virtual Work (Displacement)

- Consider the structure (or body) to be of arbitrary shape
- Suppose it is necessary to determine the displacement Δ of point A on the body caused by the "real loads" P_1, P_2 and P_3

Principle of Virtual Work (Displacement)

- Since no external load acts on the body at A and in the direction of the displacement Δ, the displacement can be determined by first placing on the body a “virtual” load suc that this force P’ acts in the same direction as Δ, (see Figure)

Principle of Virtual Work (Displacement)

- We will choose *P'* to have a unit magnitude, *P'* =1
- Once the virtual loadings are applied, then the body is subjected to the real loads P_1, P_2 and P_3, (see Figure)
- Point A will be displaced an amount Δ causing the element to deform an amount *dL*

Principle of Virtual Work (Displacement)

- As a result, the external virtual force P' & internal load *u* "ride along" by Δ and *dL* and therefore, perform external virtual work of 1. Δ on the body and internal virtual work of *u.dL* on the element

Virtual loadings

$$1.\Delta = \sum u.dL$$

Real displacement

- By choosing P' = 1, it can be seen from the solution for Δ follows directly since $\Delta = \Sigma udL$

Principle of Virtual Work (Slope)

- A virtual couple moment *M'* having a unit magnitude is applied at this point
- This couple moment causes a virtual load u_θ in one of the elements of the body

Principle of Virtual Work (Slope)

- Assuming that the real loads deform the element an amount *dL*, the rotation θ can be found from the virtual-work equation

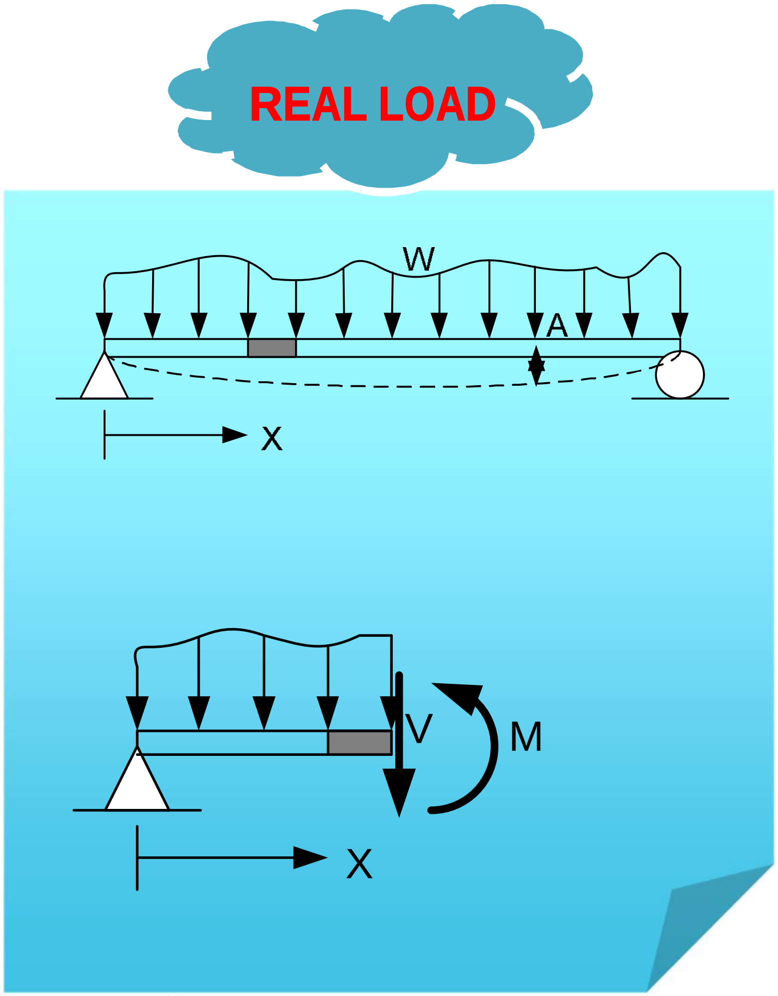
Principle of Virtual Work (Slope)

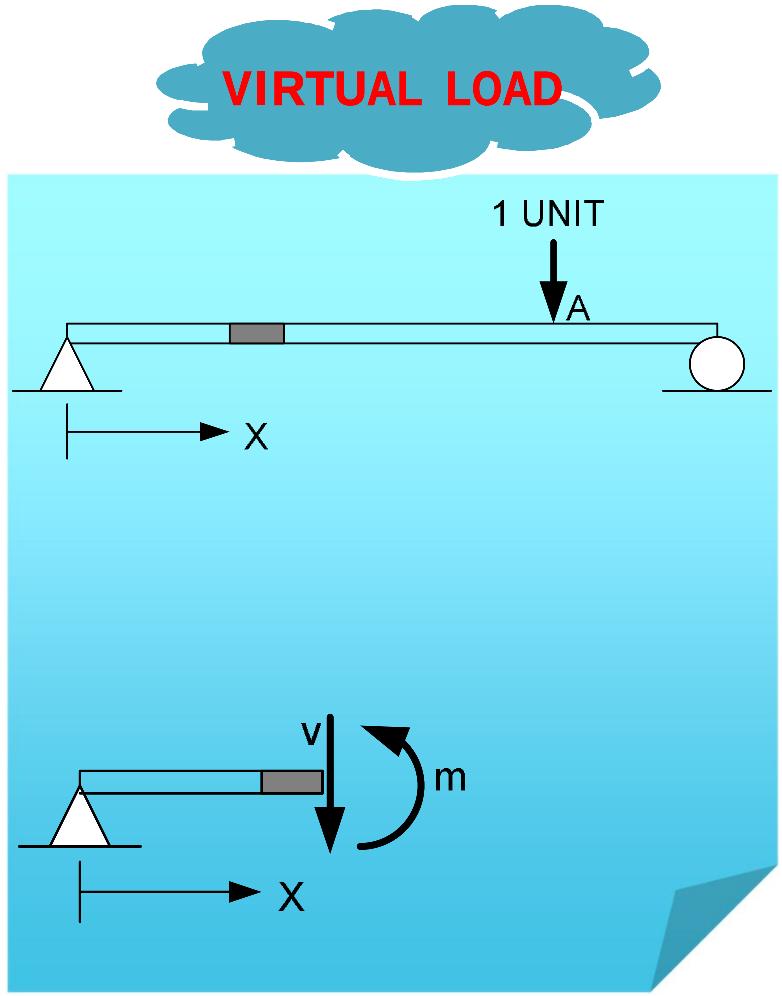
Virtual loadings

$$1.\theta = \sum u.dL$$

Real displacement

PRINCIPLE OF UNIT LOAD METHOD





- The element deform or rotate dθ = (M / EI) dx
- The external virtual work done by the unit load is 1.Δ
- The internal virtual work done by the moment, m

m dθ = m(M/EI) dx

$$1\,.\Delta = \int_0^L \frac{mM}{EI} dx$$

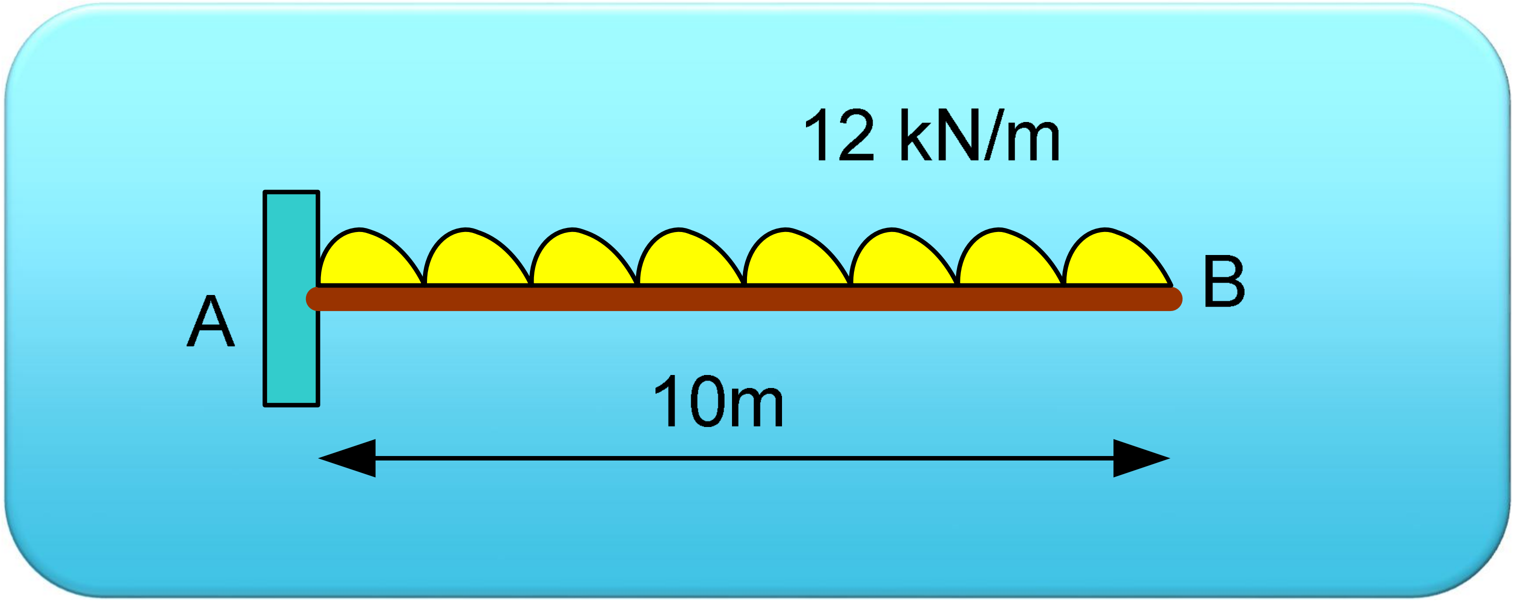
Similarly

$$1\,.\theta = \int_0^L \frac{mM}{EI} dx$$

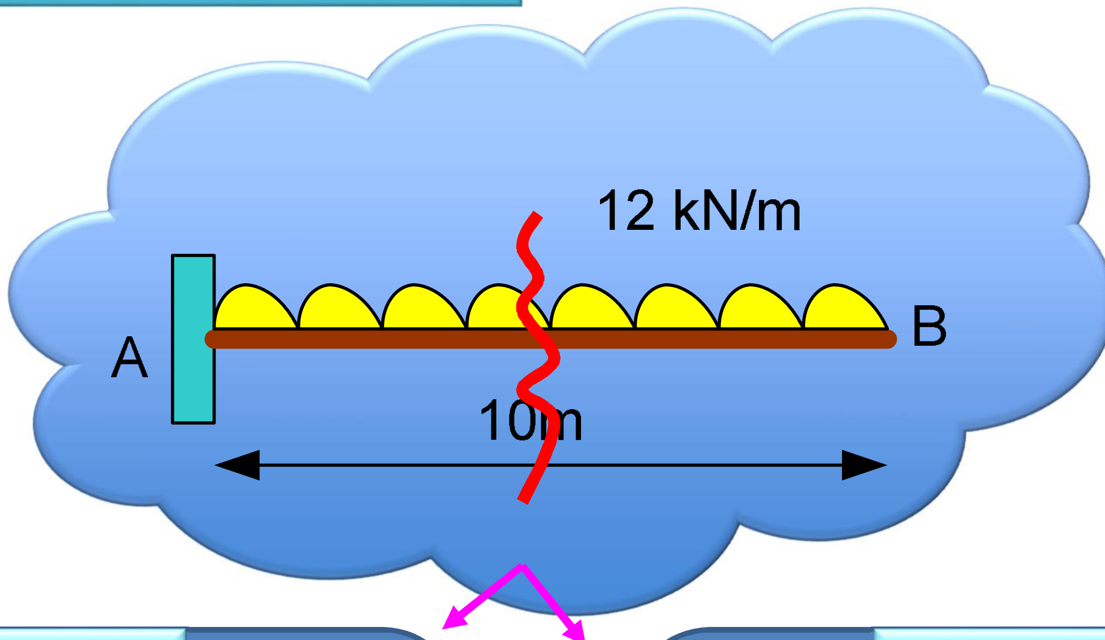
Example 1

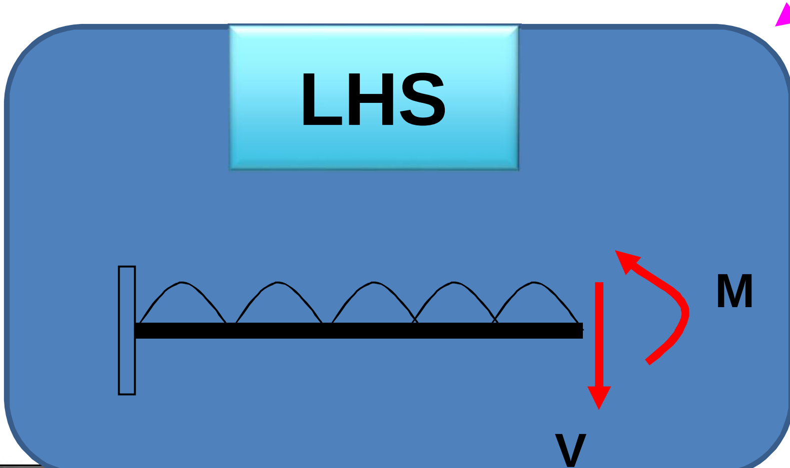
Determine the displacement at point B of the steel beam shown in figure.

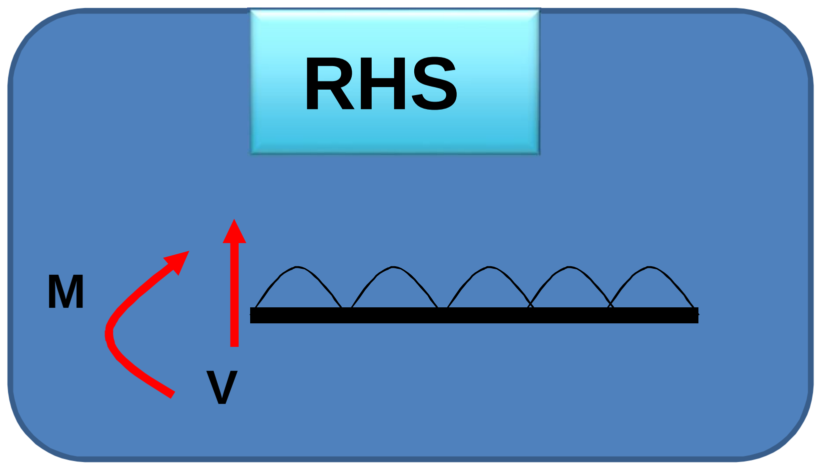
Take E = 200GPa, I = 500x 10^6 mm4



Real Moment, M

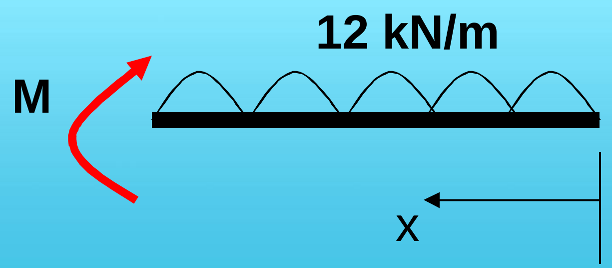






Real Moment, M

RHS

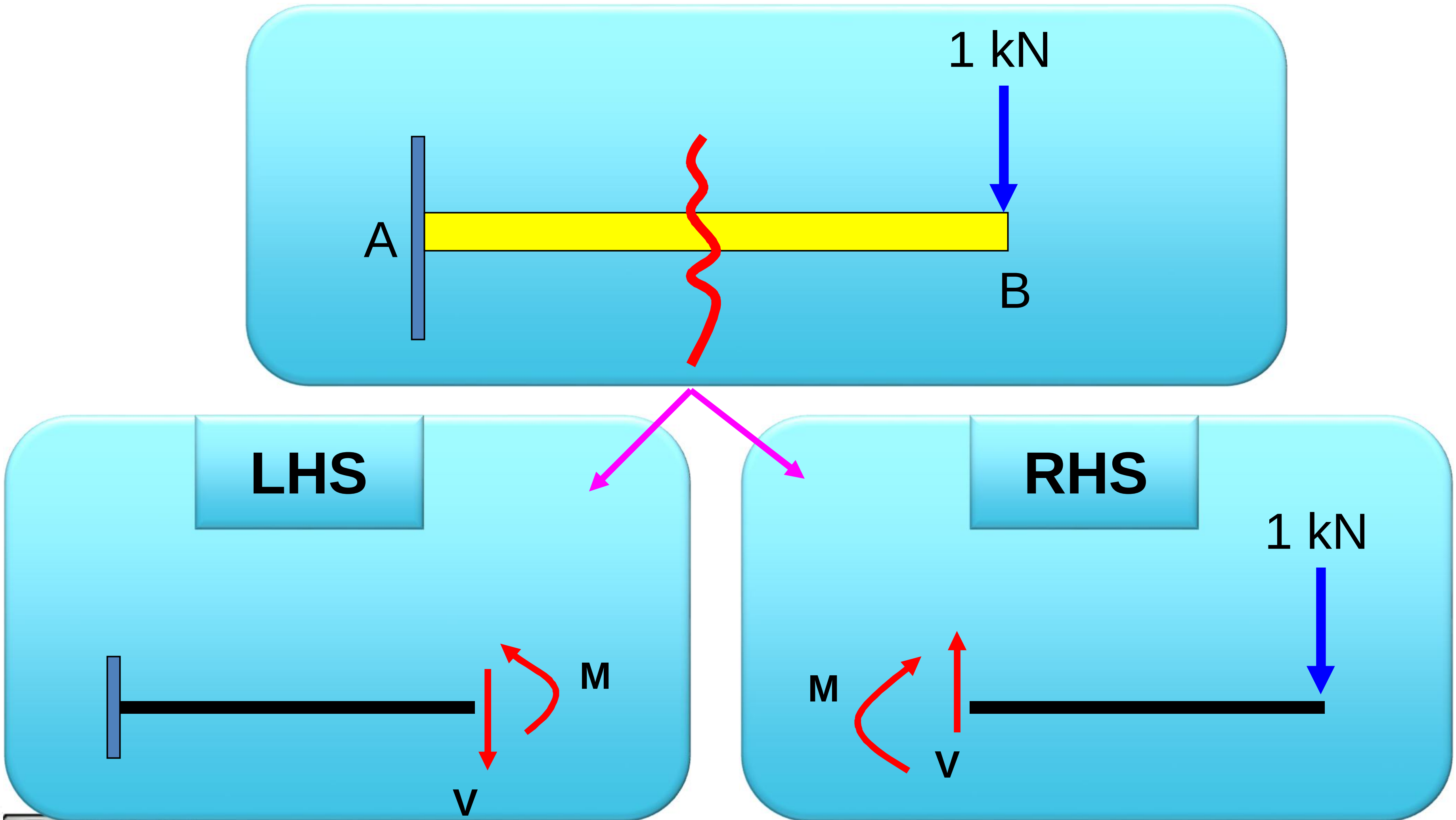


$$\sum M_x = 0 \quad (clockwise + ve)$$

$$M + \frac{12x^2}{2} = 0$$

$$M = -6x^2$$

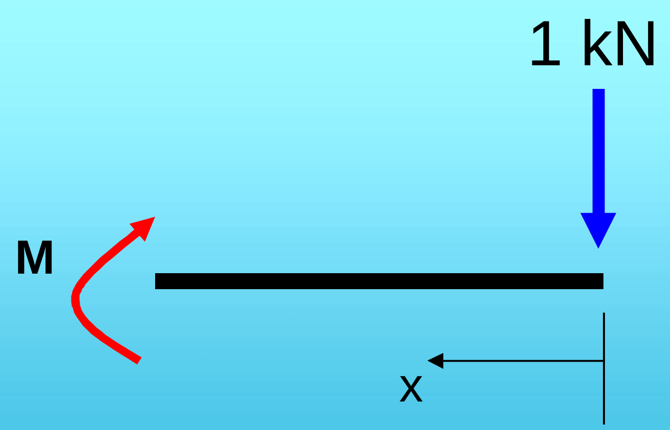
Virtual Moment, m



Virtual Moment, m

Considered RHS

$0<x<10$



$$\sum M_x = 0 \quad (clockwise + ve)$$

$$m + 1.x = 0$$

$$m = -1.x$$

Virtual-Work Equation

$$1kN.\Delta = \int_0^L \frac{mM}{EI} dx$$

$$= \int_0^{10} \frac{(-x)(-6x^2)}{EI} dx$$

$$= \frac{15\times10^3 kN.m^3}{EI}$$

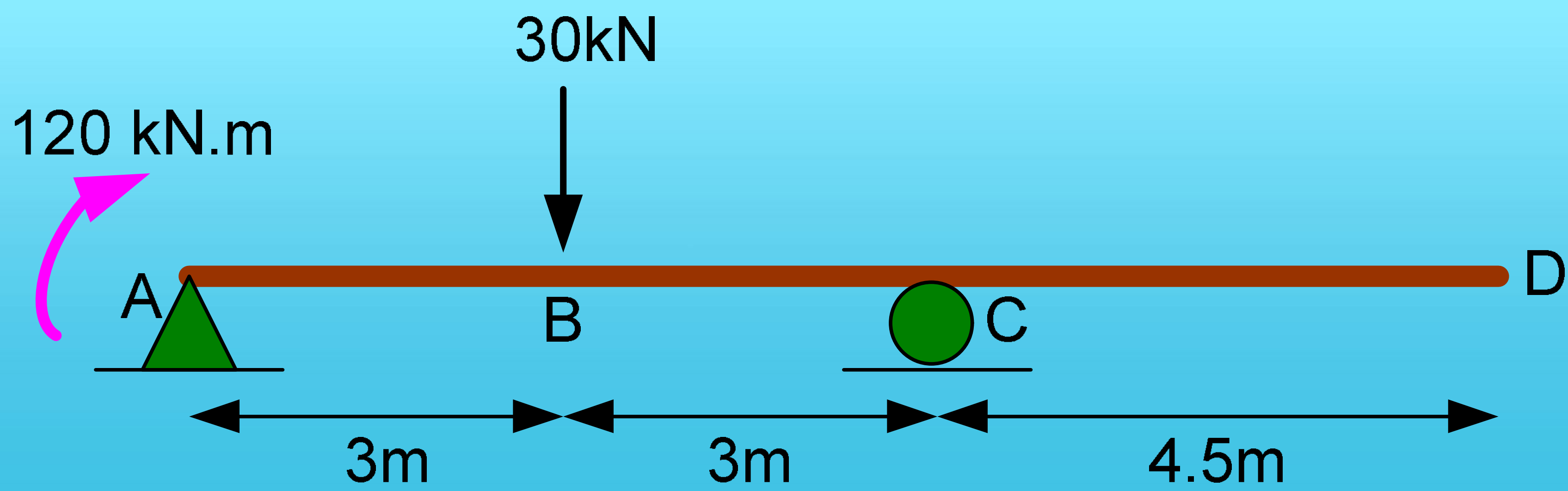
OR

$$\Delta_B = \frac{15\times10^3 kN.m^3}{200(10^6)kN/m^2(500(10^6)mm^4)(10^{-12}m^4/mm^4)}$$

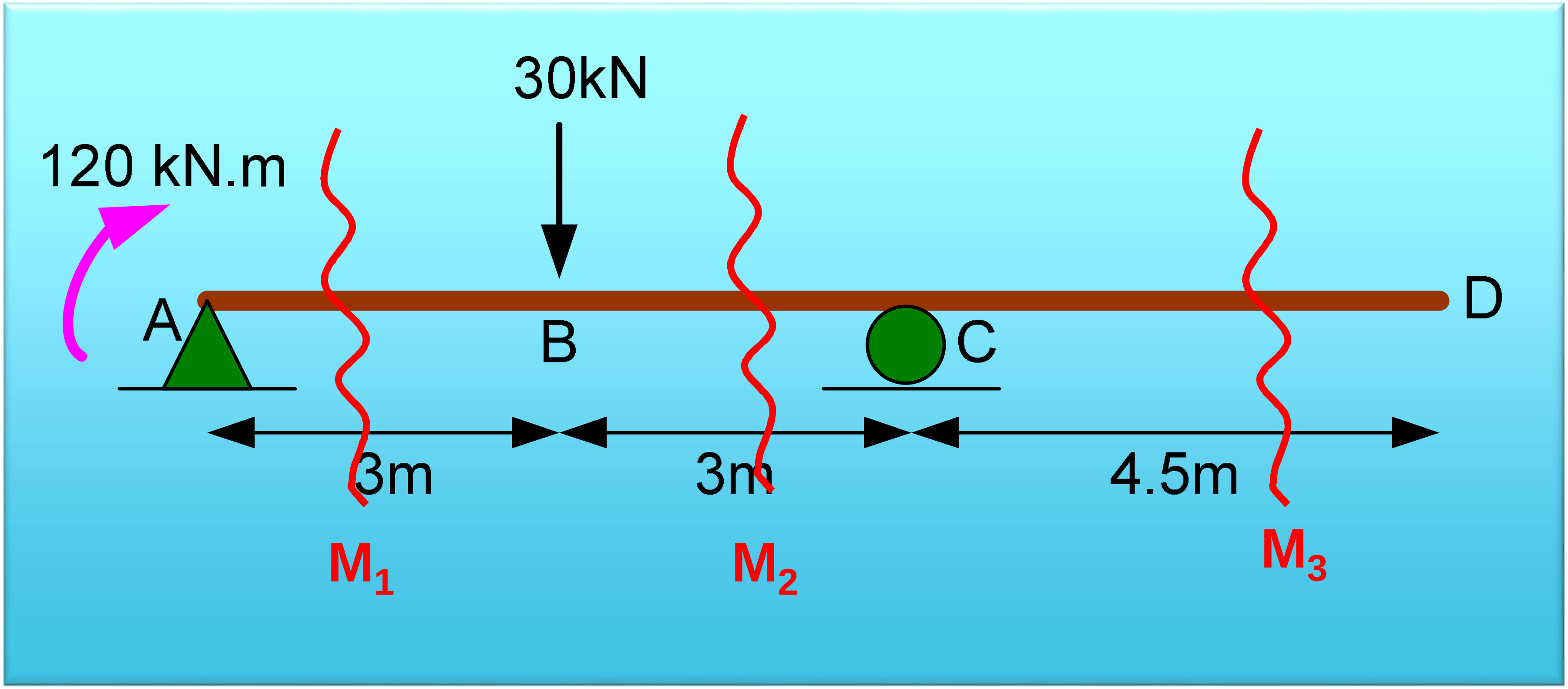
$$= 0.150m \qquad = 150mm$$

Example 2

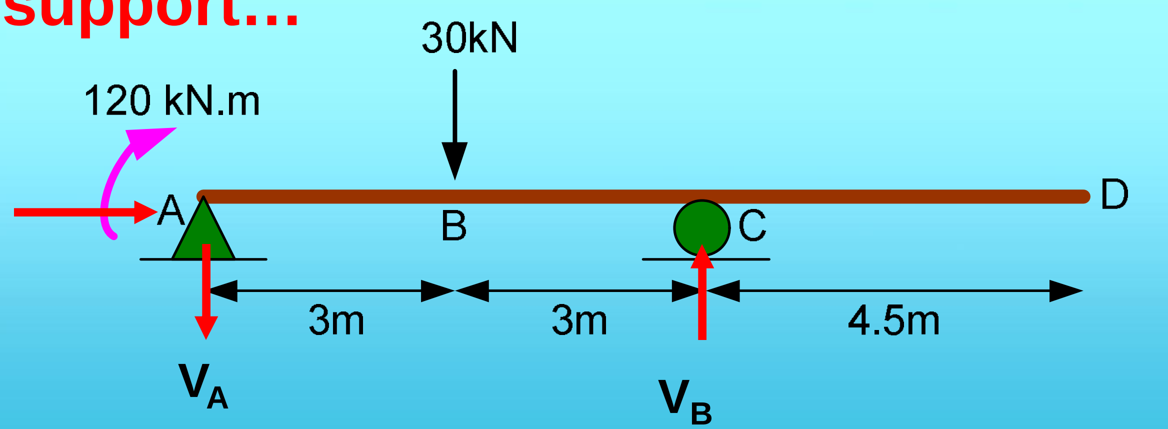
Determine the displacement at D of the steel beam in figure. Take E = 200GPa, I = 300E6 mm4



Real Moment, M



Determine the reaction at support...



$$\sum M_A = 0 \quad (clockwise + ve)$$

$$120 + 30(3) - V_B(6) = 0$$

$$V_B = 35kN$$

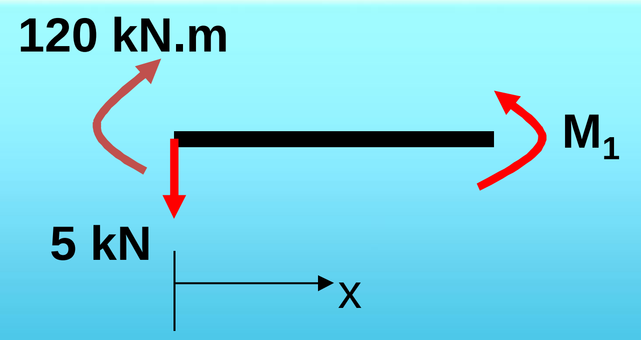
$$\sum F_y = 0 \quad (upward + ve)$$

$$-V_A + 35 - 30 = 0$$

$$V_A = 5kN$$

Member AB : LHS

$$0 \le x \le 3$$



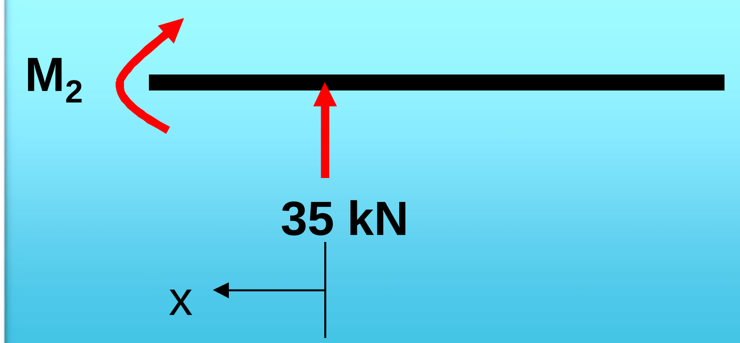
$$\sum M_x = 0 \quad (clockwise + ve)$$

$$-M_1 - 5(x) + 120 = 0$$

$$M_1 = 120 - 5x$$

Member BC : RHS

$$0 \le x \le 3$$



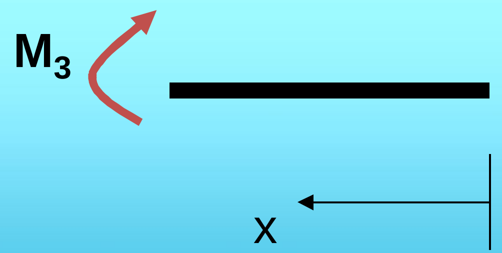
$$\sum M_x = 0 \quad (clockwise + ve)$$

$$M_2 - 35(x) = 0$$

$$M_2 = 35(x)$$

Member CD : RHS

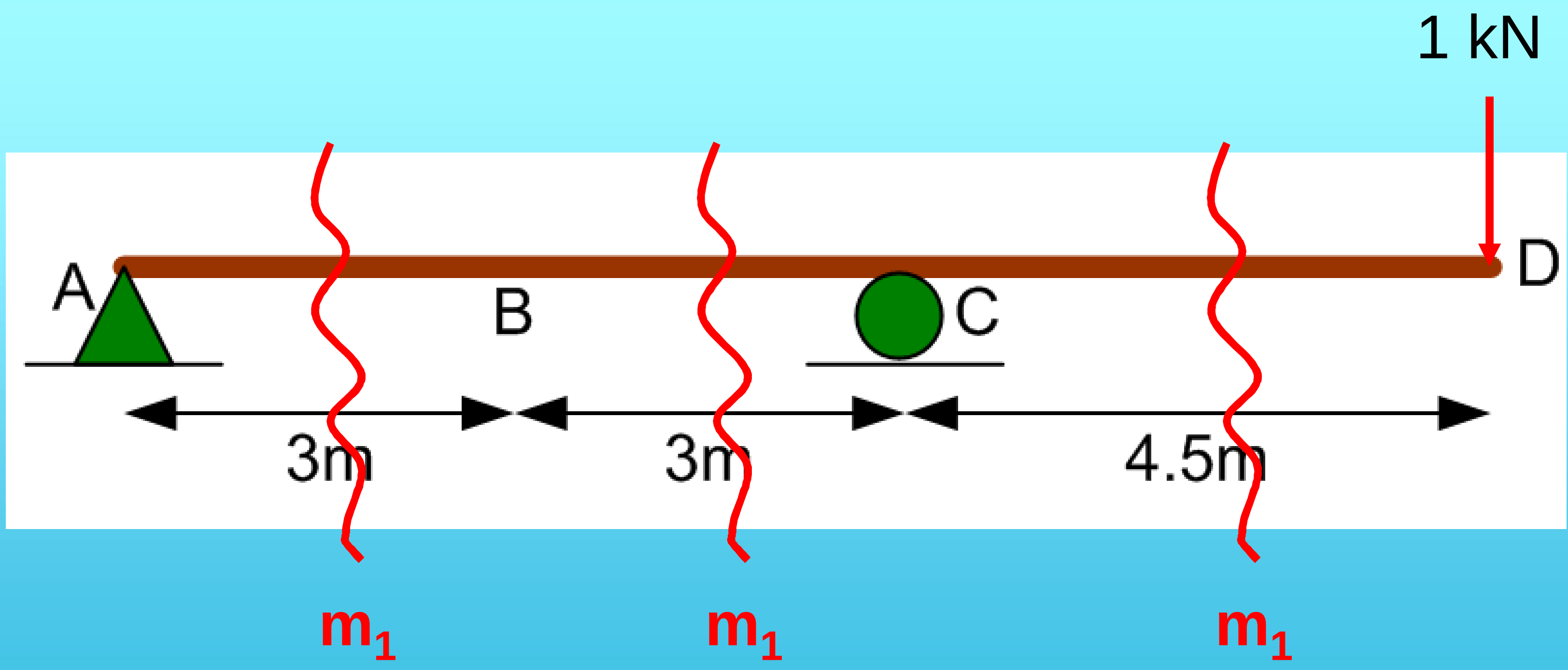
$0 \le x \le 4.5$



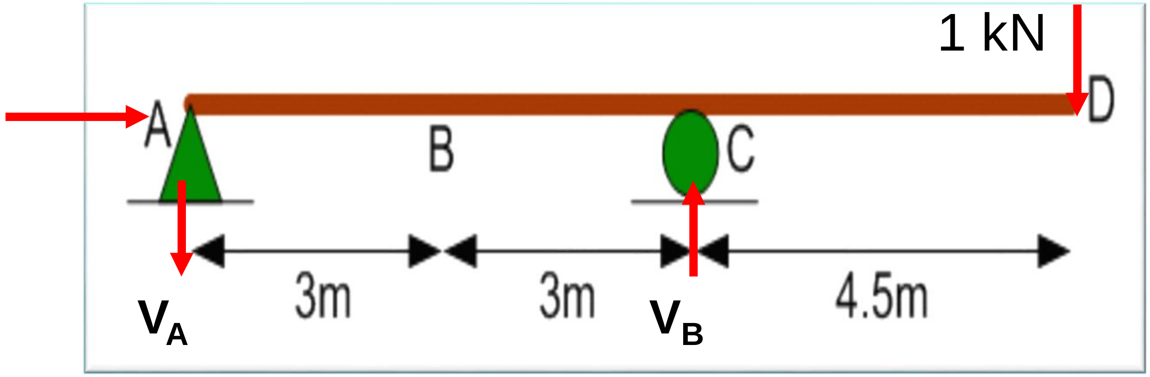
$$\sum M_x = 0 \quad (clockwise + ve)$$

$$M_3 = 0$$

Virtual moment, m



Determine the reaction at support

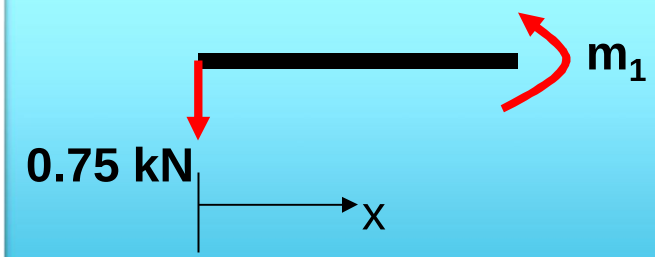


$$\sum M_A = 0 \ \ (clockwise + ve)$$
$$1(10.5) - V_B(6) = 0$$
$$V_B = 1.75kN$$

$$\sum F_y = 0 \ \ (upward + ve)$$
$$-V_A + 1.75 - 1 = 0$$
$$V_A = 0.75kN$$

Member AB : LHS

$$0 \le x \le 3$$



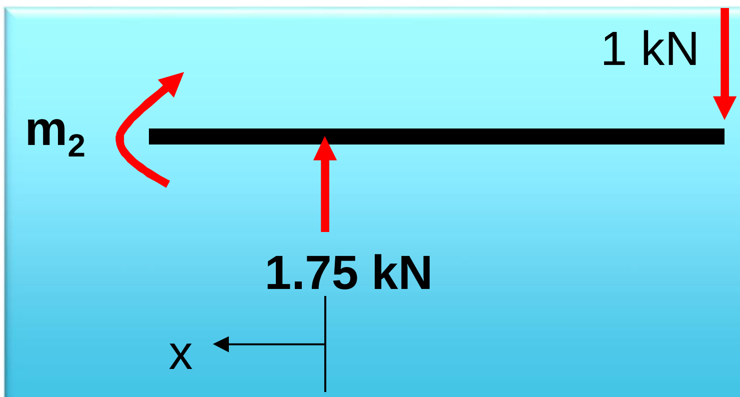
$$\sum M_x = 0 \quad (clockwise + ve)$$

$$-m_1 - 0.75(x) = 0$$

$$m_1 = -0.75x$$

Member BC : RHS

$$0 \le x \le 3$$



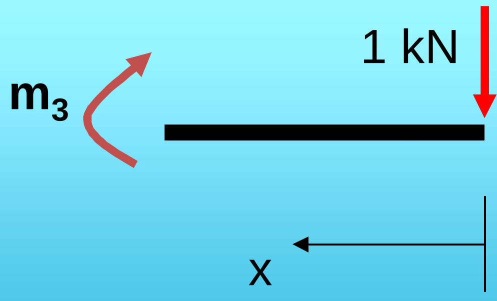
$$\sum M_x = 0 \quad (clockwise + ve)$$

$$m_2 - 1.75(x) + 1(x + 4.5) = 0$$

$$m_2 = 0.75(x) - 4.5$$

Member CD : RHS

$0 \le x \le 4.5$



$$\sum M_x = 0 \quad (clockwise + ve)$$

$$m_3 + 1.x = 0$$

$$m_3 = -x$$

Virtual-Work Equation

$$1kN.\Delta_d = \int_0^L \frac{mM}{EI}dx$$

$$= \int_0^3 \frac{m_1M_1}{EI}dx + \int_0^3 \frac{m_2M_2}{EI}dx + \int_0^{4.5} \frac{m_3M_3}{EI}dx$$

$$= \int_0^3 \frac{(-0.75x)(120-5x)}{EI}dx + \int_0^3 \frac{(0.75x-4.5)(35x)}{EI}dx$$

$$+ \int_0^{4.5} \frac{(-x)(0)}{EI}dx$$

$$\Delta_D = -\frac{371.25}{EI} - \frac{472.5}{EI} + \frac{0}{EI}$$

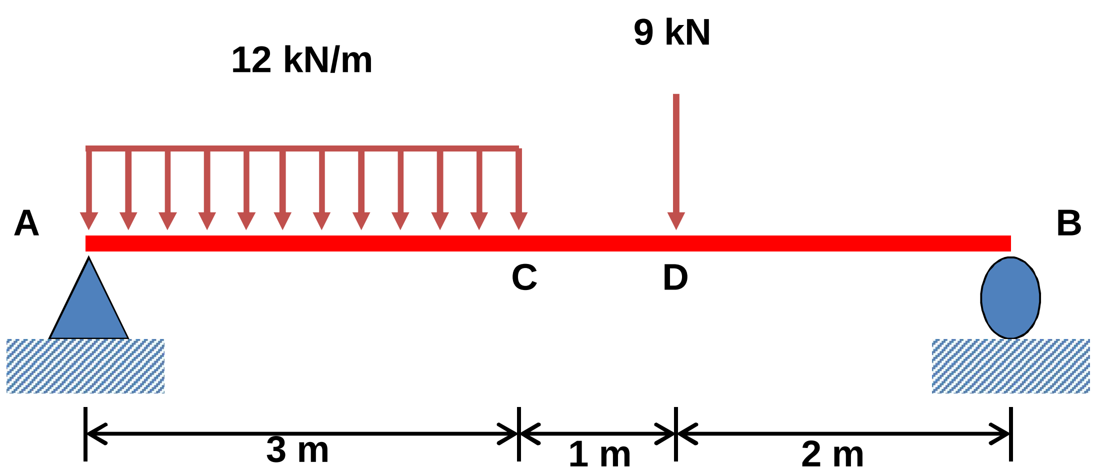
$$= -\frac{843.75\,kN.m^3}{EI}$$

$$\Delta_D = \frac{-843.75\,kN.m^3}{200(10^6)kN/m^2(300(10^6)mm^4)(10^{-12}m^4/mm^4)}$$

$$= -0.0141\,m \qquad = -14.1mm$$

Example 3

Determine the slope at A and deflection at C in the beam shown below



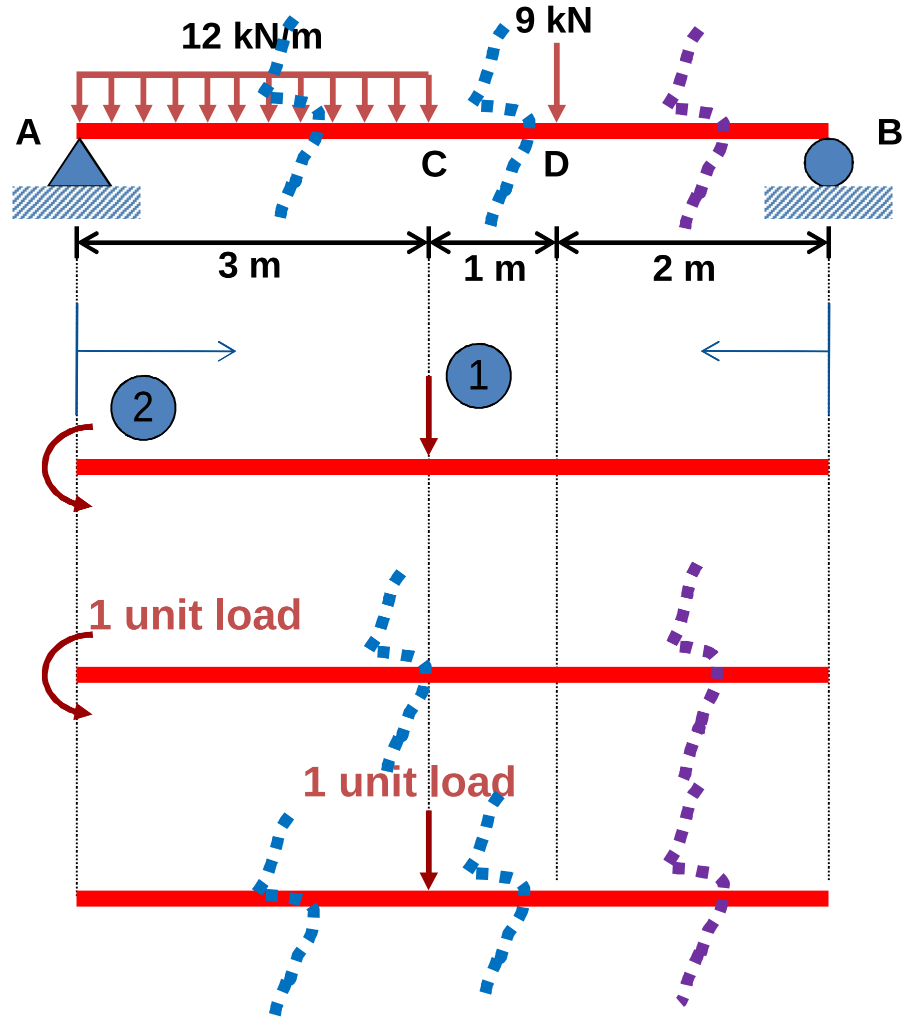
Solution

Real Load (M)

Generalized coordinates

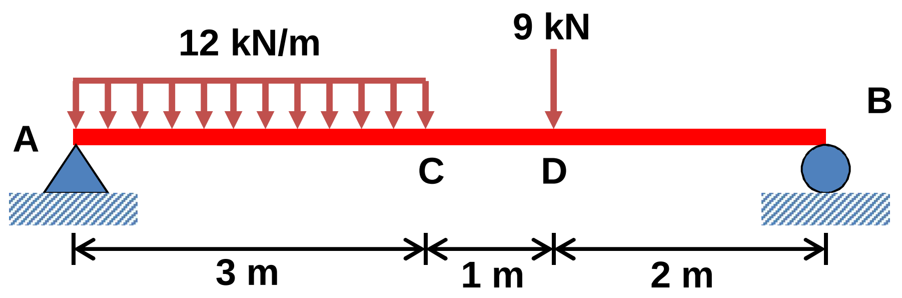
Virtual Load (m_θ): *Slope*

Virtual Load (m_Δ): *Deflection*



Real Load → M ?

1. Support reaction,



$$\sum M_A = 0 (clockwise +),$$

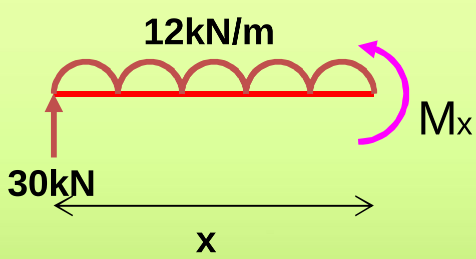
$$-R_B(6) + 9(4) + 12(3)\left(\frac{3}{2}\right) = 0$$

$$\therefore R_B = 15kN$$

$$\sum F_y \uparrow^{+} = 0,$$

$$R_A - 12(3) - 9 + 15 = 0$$

$$\therefore R_A = 30kN$$

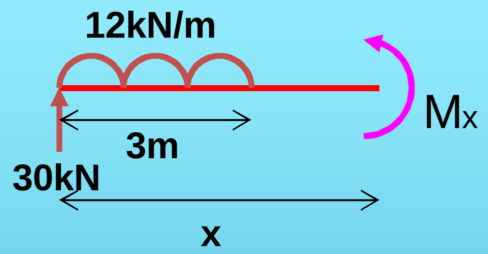


$$\text{Real Load (M)}: 0 \le x \le 3\,(segment\ AC)$$

$$\sum M_x = 0\,(clockwise+),$$

$$M_x = 30x - 12\left(\frac{x^2}{2}\right)$$

$$\therefore M_x = 30x - 6x^2 \ldots\ldots\ldots(i)$$



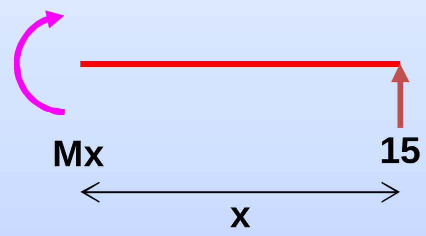
$$\text{Real Load (M)}: 3 \le x \le 4\,(segment\ CD)$$

$$\sum M_x = 0\,(clockwise +),$$

$$M_x = 30x - 12(3)(x - 3/2)$$

$$M_x = 30x - 36x + 54$$

$$\therefore M_x = -6x + 54 ....(ii)$$



$$\text{Real Load (M)}: 0 \le x \le 2\,(segment\ BD)$$

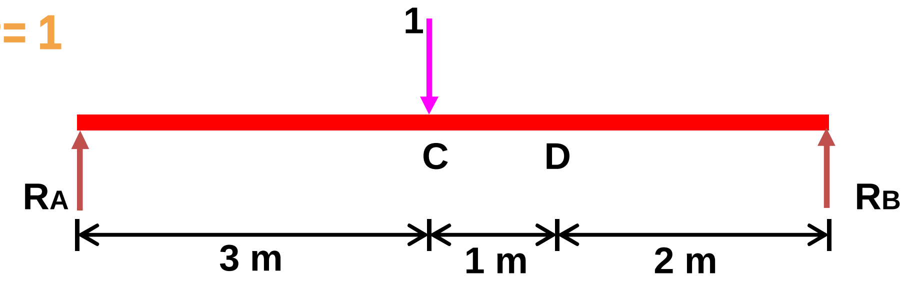
$$\sum M_x = 0\,(clockwise+),$$

$$-M_x = -15x$$

$$\therefore M_x = 15\,x.........(iii)$$

Virtual Load, m for deflection

Apply point load P= 1



$$\sum M_A = 0\,(clockwise+),$$

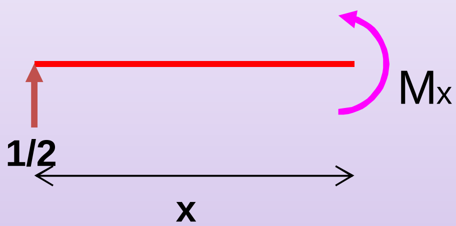
$$-R_B(6) + 1(3) = 0$$

$$\therefore R_B = \frac{1}{2}$$

$$\sum F_y \uparrow^{+} = 0,$$

$$R_A + R_B - 1 = 0$$

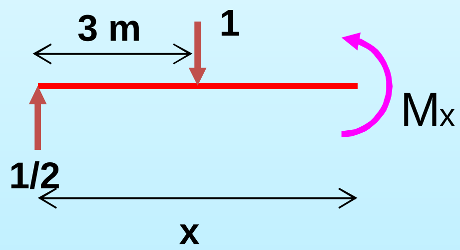
$$\therefore R_A = \frac{1}{2}$$



$$\text{Virtual Load (m)}: 0 \le x \le 3(segment\ AC)$$

$$\sum M_x = 0(clockwise+),$$

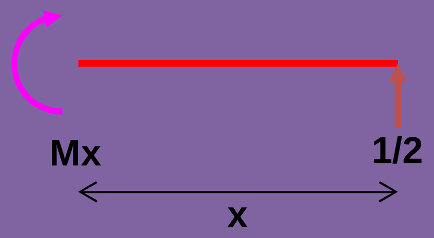
$$\therefore M_x = \frac{1}{2}x.........(i)$$



$$\text{Virtual Load (m)}: 3 \le x \le 4\,(segment\ CD)$$

$$\sum M_x = 0\,(clockwise +),$$

$$\therefore M_x = -0.5x + 3 \ldots\ldots\ldots(ii)$$



$$\text{Virtual Load (m)} : 0 \le x \le 2\,(segment\ BD)$$

$$\sum M_x = 0\,(clockwise +),$$

$$-M_x = -\frac{1}{2}x$$

$$\therefore M_x = \frac{1}{2}x \ldots\ldots\ldots(iii)$$

$$\text{Deflection at } C, \Delta_D :$$

$$\Delta_D = \int \frac{Mm}{EI} dx$$

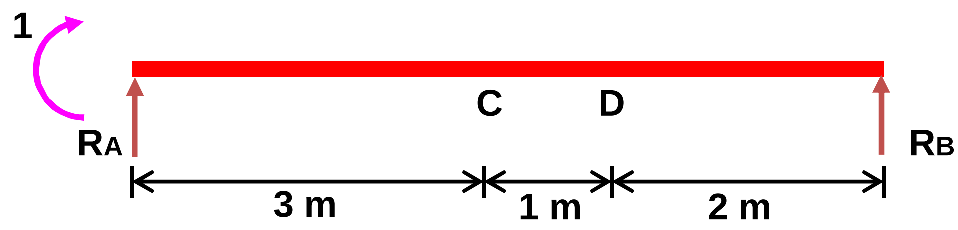
$$= \frac{1}{EI}\int_0^3 (30x - 6x^2)(0.5x)dx + \frac{1}{EI}\int_3^4 (-6x + 54)(-0.5x + 3)dx$$

$$\frac{1}{EI}\int_0^2 (15x)(0.5x)dx$$

$$= \frac{135.75}{EI}$$

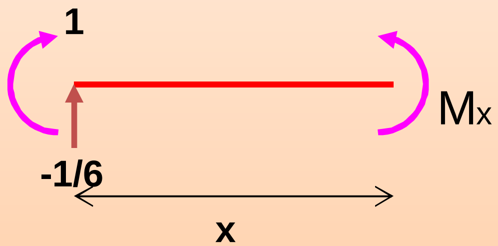
Virtual Load, m for rotation

Apply $m_\theta = 1$



$$\sum M_A = 0 (clockwise+),$$
$$-R_B(6) + 1 = 0$$
$$\therefore R_B = \frac{1}{6}$$

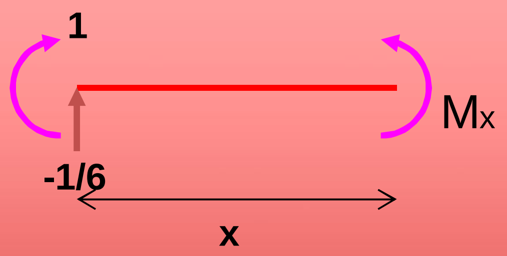
$$\sum F_y \uparrow^{+} = 0,$$
$$R_A + R_B = 0$$
$$\therefore R_A = -\frac{1}{6}$$



$$\text{Virtual Load (m)}: 0 \le x \le 3(\textit{segment AC})$$

$$\sum M_x = 0(\textit{clockwise}+),$$

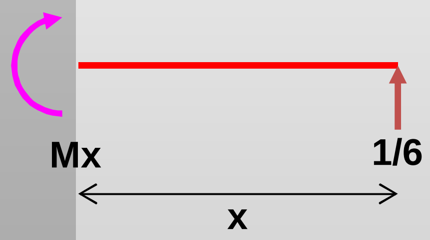
$$\therefore M_x = 1 - \frac{1}{6}x..........(i)$$



$$\text{Virtual Load (m)}: 3 \le x \le 4\,(\textit{segment CD})$$

$$\sum M_x = 0\,(\textit{clockwise}+),$$

$$\therefore M_x = 1 - \frac{1}{6}x \ldots\ldots\ldots(ii)$$



$$\text{Virtual Load (m)} : 0 \le x \le 2\,(segment\ BD)$$

$$\sum M_x = 0\,(clockwise +),$$

$$-M_x = -\frac{1}{6}x$$

$$\therefore M_x = \frac{1}{6}x.........(iii)$$

Slope at A, θ_A :

$$\theta_A = \int \frac{Mm}{EI} dx$$

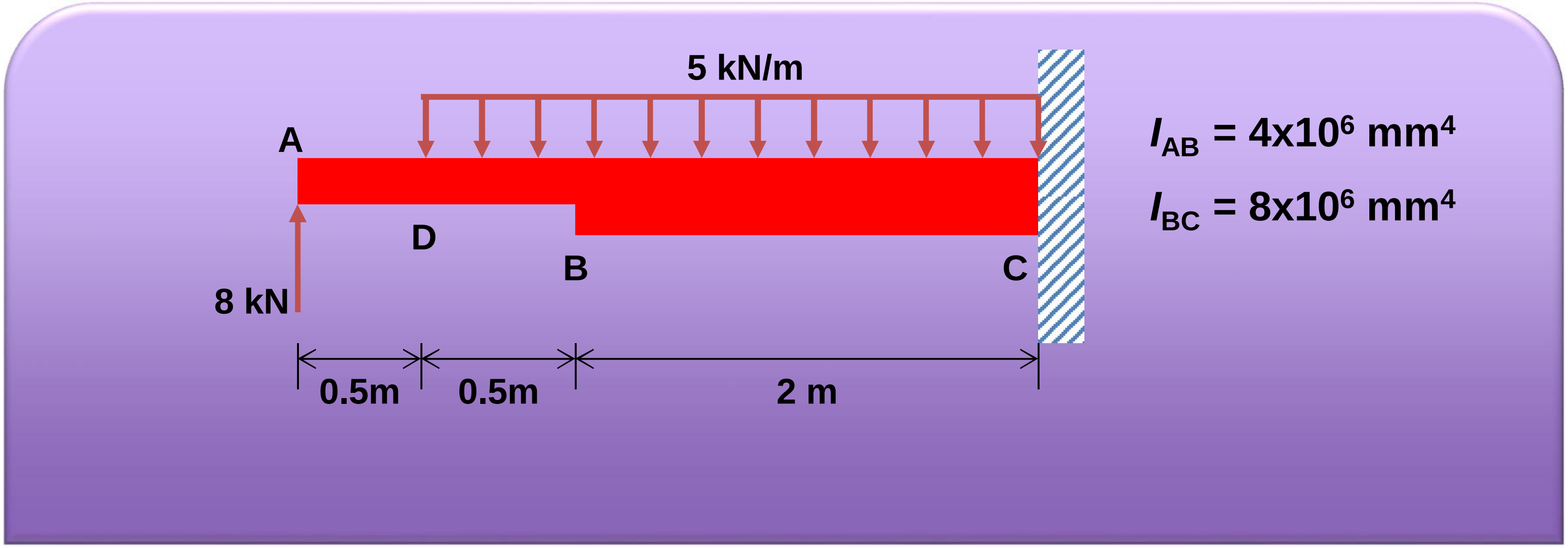
$$= \frac{1}{EI}\int_0^3 (30x - 6x^2)\left(1 - \frac{x}{6}\right)dx + \frac{1}{EI}\int_3^4 (-6x + 54)\left(1 - \frac{x}{6}\right)dx$$

$$\frac{1}{EI}\int_0^2 (15x)\left(\frac{x}{6}\right)dx$$

$$= \frac{76.75}{EI}$$

Example 4

Determine the slope and deflection at B in the beam shown below. Given E=200 kN/mm²



Moment equation (deflection):

Segment	Condition	*I* mm^4	m (deflection)	M
AD	0<x<0.5	4×10^6	0	8x
DB	0.5<x<1	4×10^6	0	$8x – 2.5 (x – 0.5)^2$
BC	1<x<3	8×10^6	x – 1	$8x – 2.5 (x- 0.5)^2$

$Deflection, \Delta_B$

$$\Delta_B = \int \frac{mM}{EI} dx$$

$$= \int_1^3 \frac{(x-1)(-2.5x^2+10.5x-0.625)}{(200\times 10^6)(8\times 10^{-6})} dx$$

$$= \frac{1}{1600}\left[-\frac{2.5x^4}{4}+\frac{13x^3}{3}-\frac{11.125x^2}{2}+0.625x\right]_1^3$$

$$= 0.012m$$

$$= 12mm$$

Moment equation (slope):

Segment	Condition	I mm^4	m (slope)	M
AD	0<x<0.5	4×10^6	0	8x
DB	0.5<x<1	4×10^6	0	$8x – 2.5 (x – 0.5)^2$
BC	1<x<3	8×10^6	– 1	$8x – 2.5 (x- 0.5)^2$

$$Slope, \theta_B$$

$$\theta_B = \int \frac{mM}{EI} dx$$

$$= \int_1^3 \frac{(1)(-2.5x^2 + 10.5x - 0.625)}{(200 \times 10^6)(8 \times 10^{-6})} dx$$

$$= \frac{1}{1600} \left[-\frac{2.5x^3}{3} + \frac{10.5x^2}{2} - 0.625x \right]_1^3$$

$$= \frac{19.1}{1600}$$

$$= 0.0119 rad$$

THANKS

Author Information

Mohd Arif Bin Sulaiman
Mohd Faizal Bin Md. Jaafar
Mohammad Amirulkhairi Bin Zubir
Rokiah Binti Othman
Norhaiza Binti Ghazali
Shariza Binti Mat Aris