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THEORY OF STRUCTURES
CHAPTER 2 : DEFLECTION (MACAULAY METHOD)
PART 1

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Chapter 2 : Part 1 – Macaulay Method

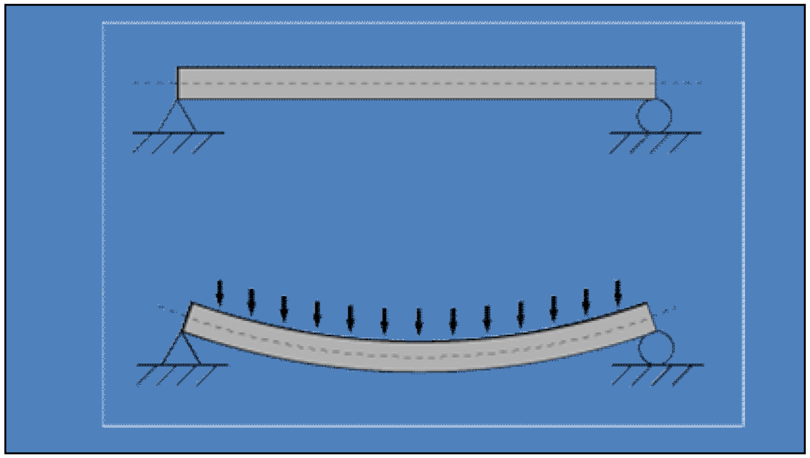
- Aims
  - Draw elastic curve for beam
  - Write equation for bending moment
  - Determine the deflection of statically determinate beam by using Double Integration Method.
  - Write a single equation for bending moment.
  - Determine the deflection of statically determinate beam by using Macaulay's Method.

- Expected Outcomes :
  - Able to analyze determinate beam – deflection and slope by Macaulay Method.

- References

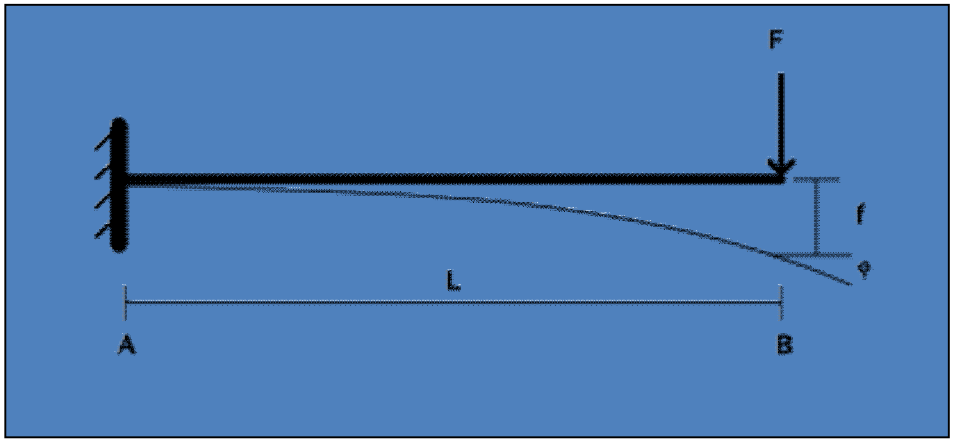
  - Mechanics of Materials, R.C. Hibbeler, 7th Edition, Prentice Hall
  - Structural Analysis, Hibbeler, 7th Edition, Prentice Hall
  - Structural Analysis, SI Edition by Aslam Kassimali,Cengage Learning
  - Structural Analysis, Coates, Coatie and Kong
  - Structural Analysis - A Classical and Matrix Approach, Jack C. McCormac and James K. Nelson, Jr., 4th Edition, John Wiley

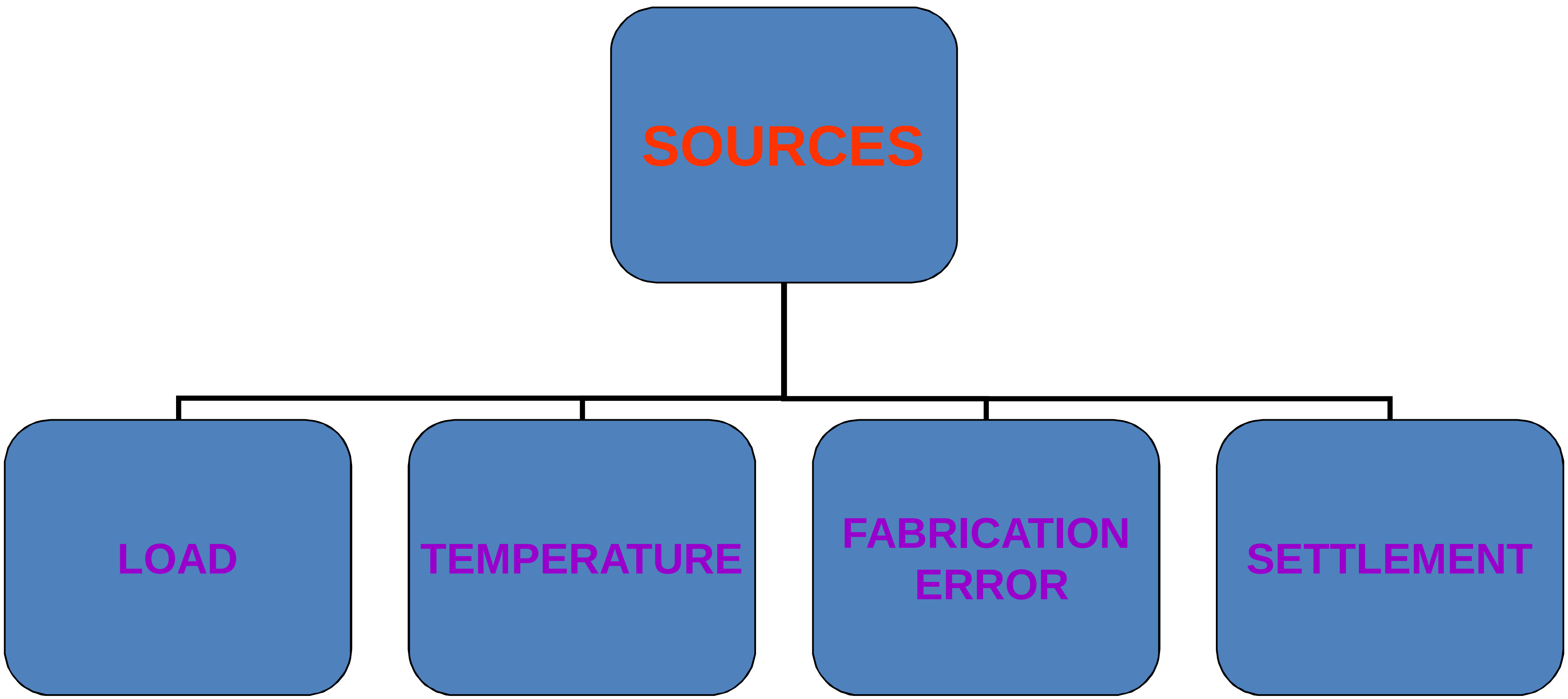
WHAT IS DEFLECTION????



INTRODUCTION

- **deflection** is a term that is used to describe the degree to which a structural element is displaced under a load.




SOURCES
LOAD
TEMPERATURE
FABRICATION ERROR
SETTLEMENT

THE ELASTIC CURVE

-The deflection diagram of the longitudinal axis that passes through the centrfold of each cross-sectional area of the beam

- Support that resist a force, such as **pinned**, **restrict displacement**

- Support that resist a moment such as **fixed**, **resist rotation or slope as well as displacement**.

Example

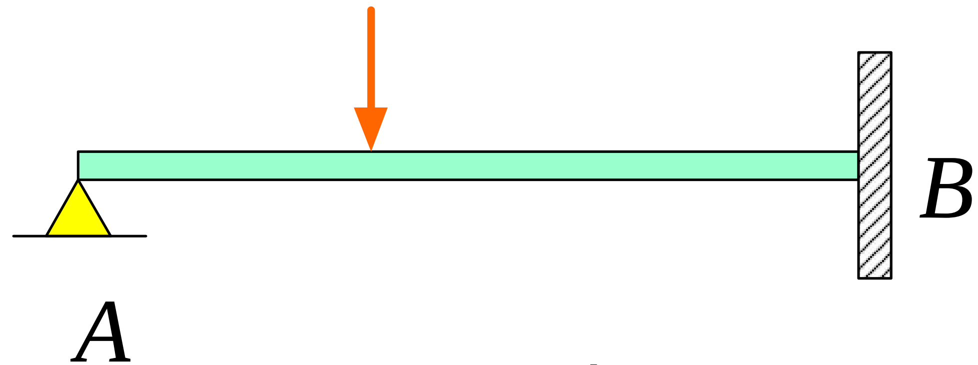


Figure 1

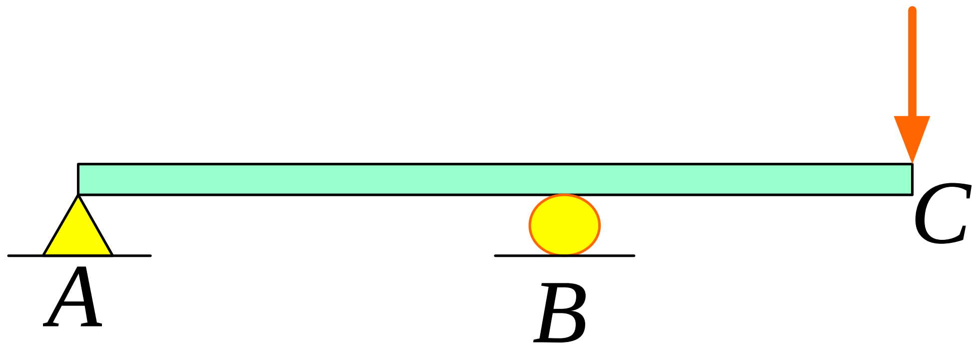


Figure 2

Example

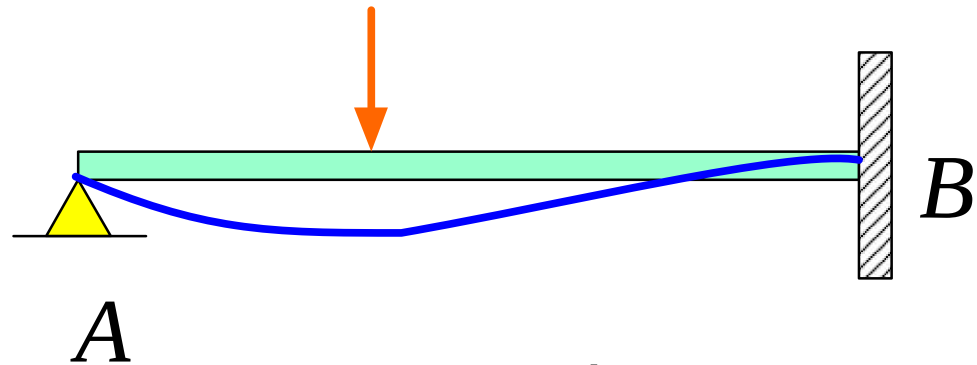


Figure 1

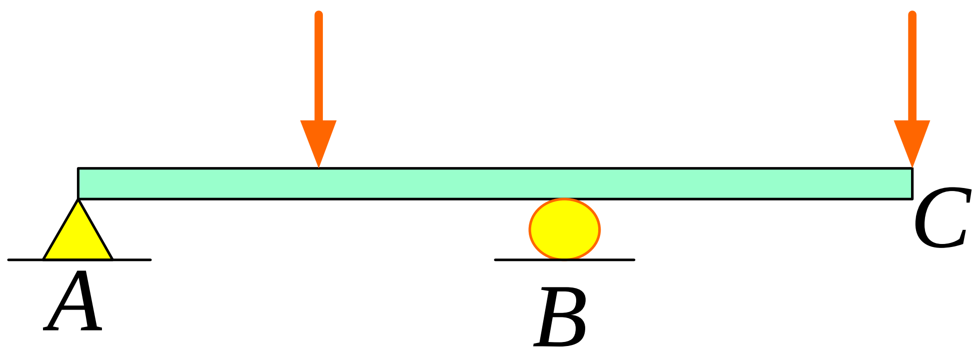


Figure 2

Example

$\theta_A \neq 0$

$\Delta_A = 0$

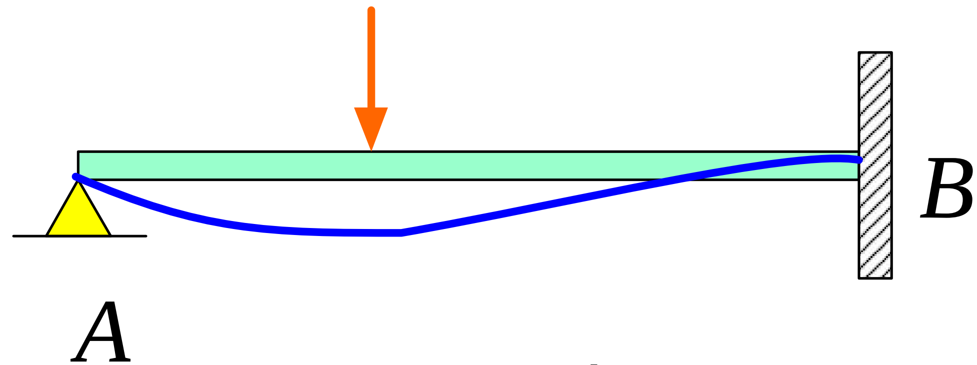


Figure 1

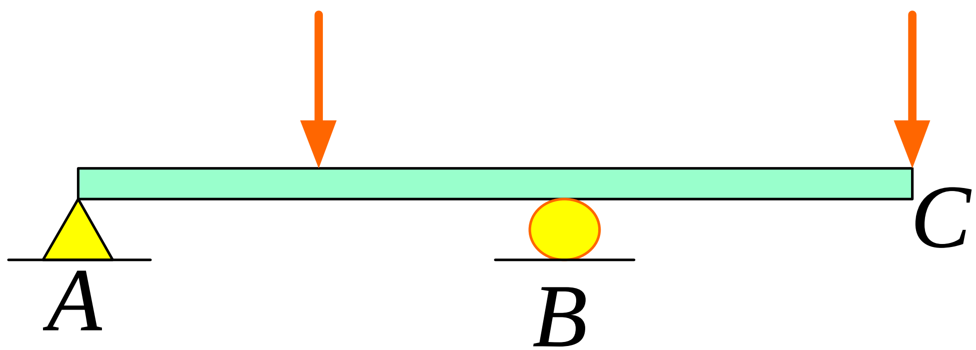


Figure 2

Example

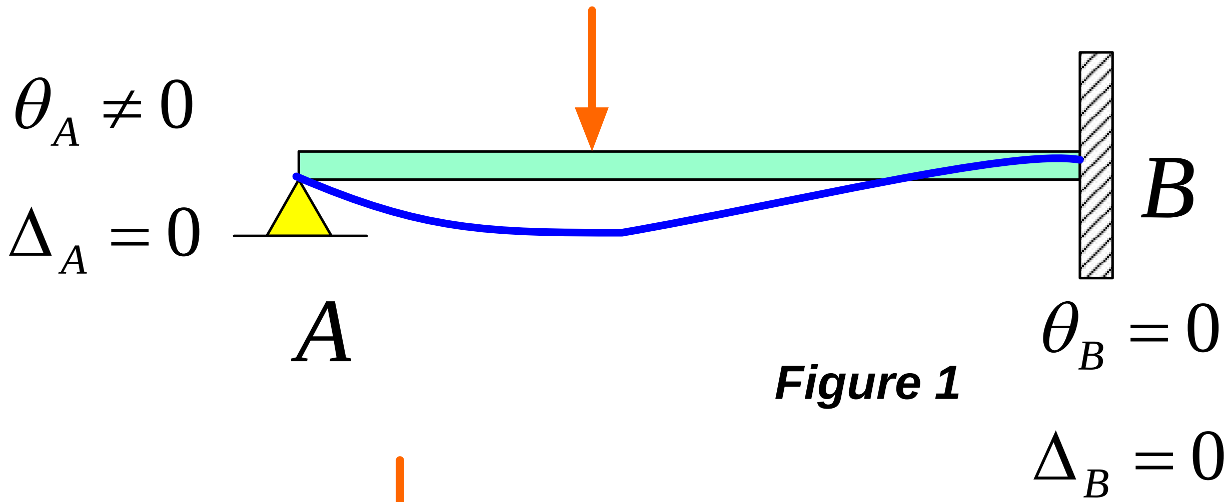


Figure 1

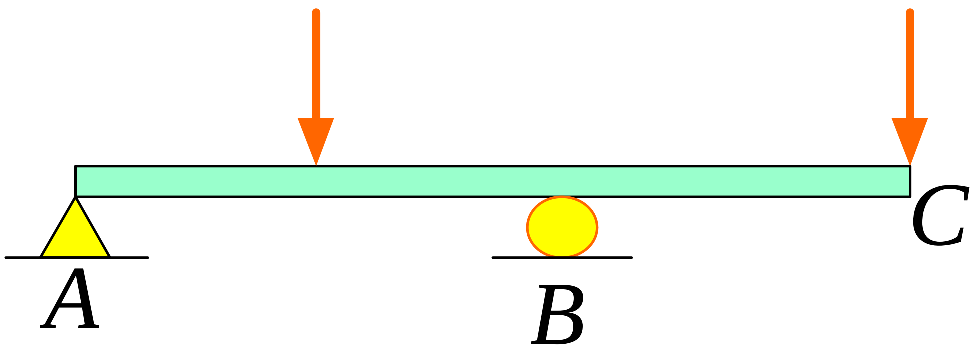


Figure 2

Example

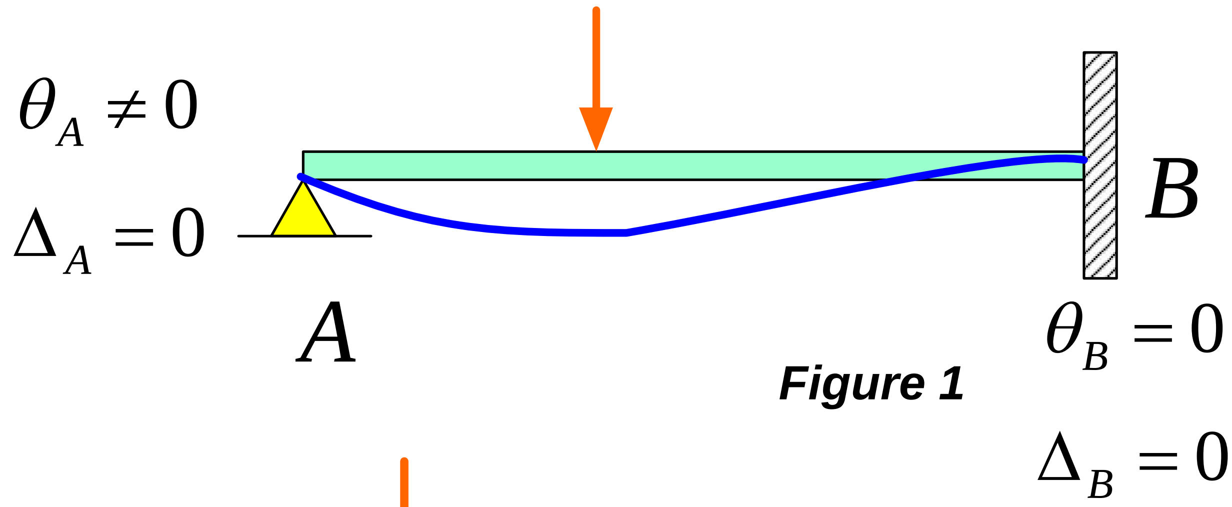


Figure 1

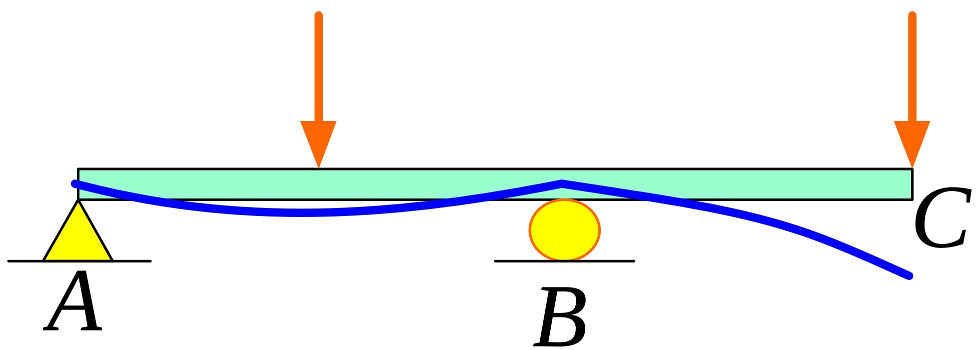


Figure 2

Example

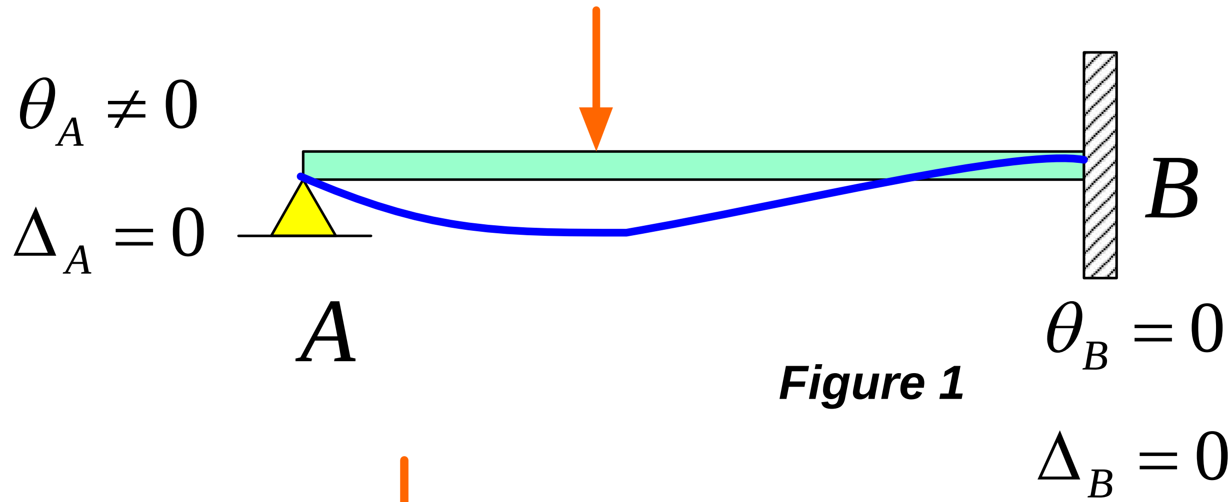


Figure 1

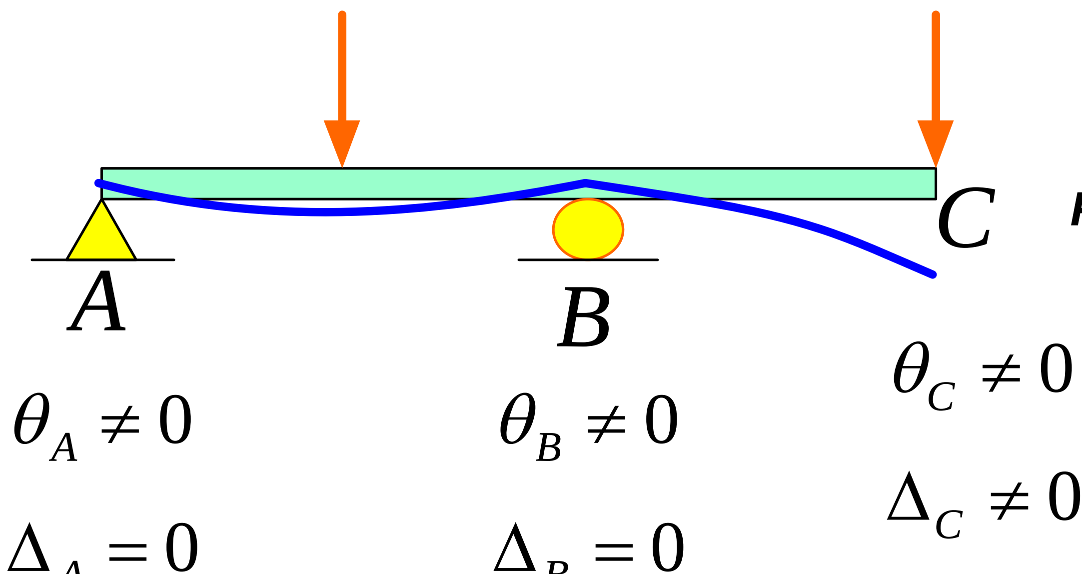
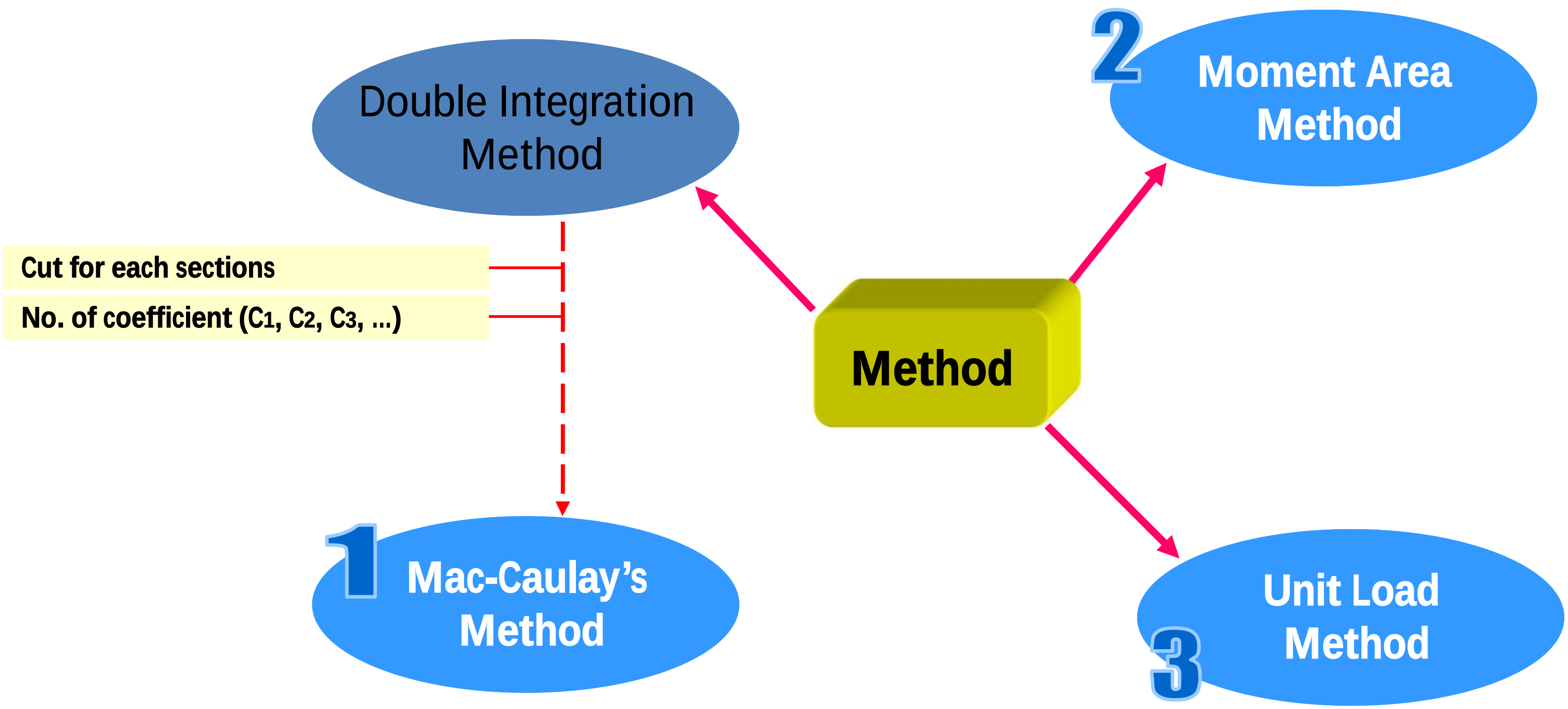


Figure 2

<u>Three</u> basic methods to find deflection for statically determinate beams:



EULER-BERNOULLI THEORY

- Also known as elastic-beam theory
- This theory form important differential equation that relate the internal moment in a beam to the displacement and slope of its elastic curve.
- This equation form the basis for the deflection methods.

$$\frac{d^2v}{dx^2} = \frac{M}{EI} \qquad \text{Equation 1}$$

THE DOUBLE INTERGRATION METHOD

- Moment, M is known expressible as a function of position x, the successive integrations of Eq. 1 will yield the beam's slope, θ.

$$\theta = \frac{dv}{dx} = \int \frac{M}{EI} dx$$

- And the equation of the elastic curve, v(displacement)

$$v = f(x) = \int \int \frac{M}{EI} dx$$

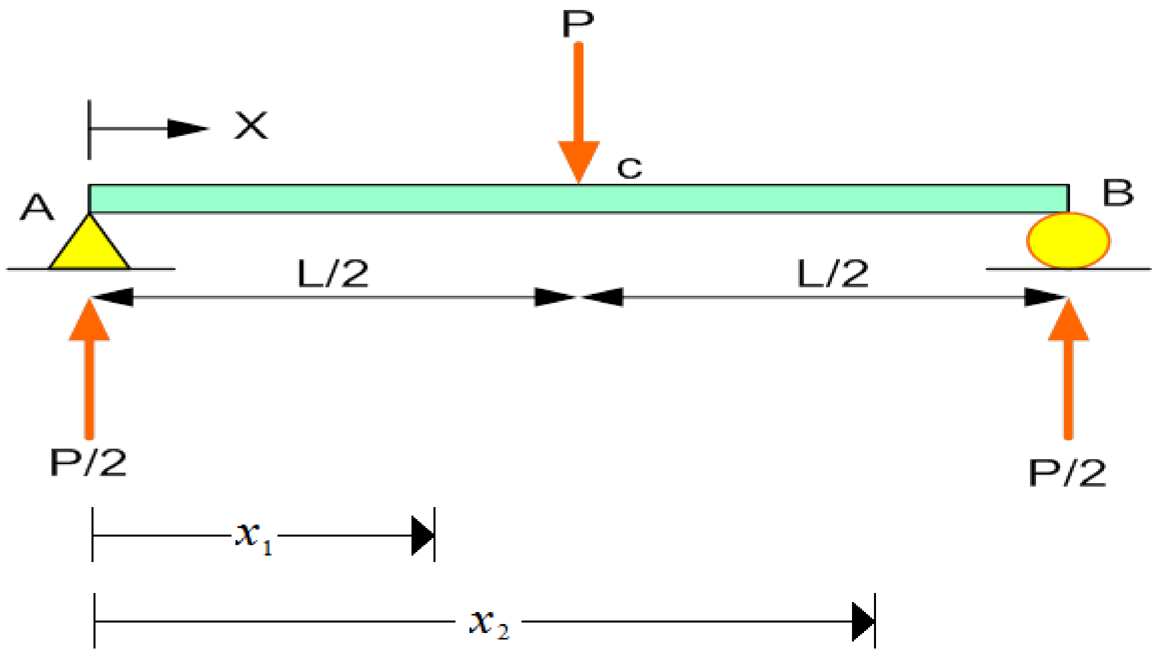
THE DOUBLE INTERGRATION METHOD

- This method depend on the loading of the beam.
- All function for moment must be written each valid within the region between discontinuities.
- Using equation 1 and the function for M, will give the slope and deflection for each region of the beam for which they are valid.

THE DOUBLE INTERGRATION METHOD

- EXAMPLE

Consider a simply supported beam AB of length L and carrying concentrated load P at mid span,C as shown below. Use the double integration method. Find the equation of the elastic curve. EI is constant.



- Function for the beam moment
- For span $0<x_1<L/2$

$$M_1 = \frac{P}{2}x_1$$

- For span $L/2<x_2<L$

$$M_2 = \frac{P}{2}x_2 - P(x_2 - \frac{L}{2})$$

$$M_2 = -\frac{P}{2}x_2 + \frac{PL}{2}$$

- Replace M_1 into slope and displacement integration.

$$\frac{d^2v}{dx^2} = \frac{\frac{P}{2}x_1}{EI}$$

$$\theta_1 EI = \frac{dv}{dx} = \int \frac{P}{2}x_1 dx$$

$$v_1 EI = f(x) = \int\int \frac{P}{2}x_1 dx$$

$$\theta_1 EI = \int \frac{P}{2} x_1 \, dx$$

$$\theta_1 EI = \frac{P x_1^2}{4} + C_1$$

Then, $$v_1 EI = \int \frac{P x_1^2}{4} + C_1 \, dx$$

$$v_1 EI = \frac{P x_1^3}{12} + C_1 x_1 + C_2$$

- Here we have 2 unknown C_1 and C_2

- Replace M_2 into slope and displacement integration.

$$\frac{d^2v}{dx^2} = \frac{-\frac{P}{2}x_2 + \frac{PL}{2}}{EI}$$

$$\theta_2 EI = \frac{dv}{dx} = \int -\frac{P}{2}x_2 + \frac{PL}{2}dx$$

$$v_2 EI = f(x) = \int\int -\frac{P}{2}x_2 + \frac{PL}{2}dx$$

$$\theta_2 EI = \int -\frac{P}{2}x_2 + \frac{PL}{2}dx$$

$$\theta_2 EI = -\frac{P}{4}x_2^2 + \frac{PL}{2}x_2 + C_3$$

Then,

$$v_2 EI = \int -\frac{P}{4}x_2^2 + \frac{PL}{2}x_2 + C_3 dx$$

$$v_2 EI = -\frac{P}{12}x_2^3 + \frac{PL}{4}x_2^2 + C_3 x_2 + C_4$$

- Here we have 2 unknown C_3 and C_4

- Using boundary conditions
  - $v_1 = 0, x_1 = 0$
  - $v_2 = 0, x_2 = L$
  - $x_1 = x_2 = \frac{L}{2}$
    - $v_1 = v_2$
    - $\theta_1 = \theta_2$
- This will solve C_1, C_2, C_3 and C_4

Solving all the unknown, C_x, will give slope and displacement for the element.

– At the support $x_1 = 0, x_2 = L$

$$\theta_1 = \theta_2 = \pm\frac{PL^2}{16}$$

– At the mid span $x_1 = x_2 = \frac{L}{2}$

$$v_1 = v_2 = -\frac{PL^3}{48}$$

MACAULAY'S METHOD

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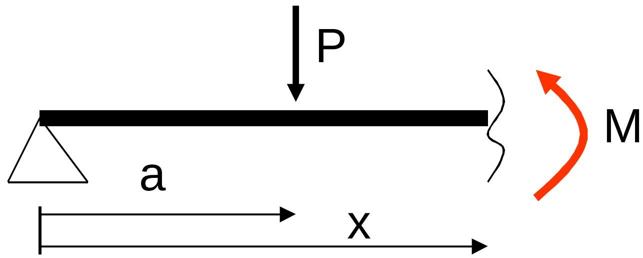
GENERAL

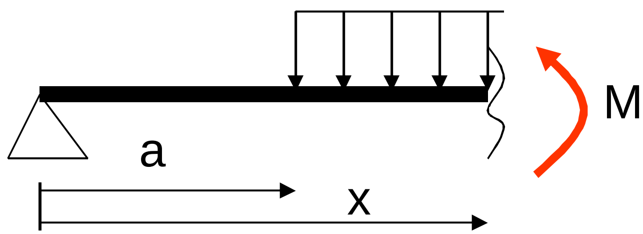
- ✓ Mac-Caulay's method is a means to find the equation that describes the deflected shape of a beam
- ✓ From this equation, any deflection of interest can be found
- ✓ Mac-Caulay's method enables us to write a single equation for bending moment for the full length of the beam
- ✓ When coupled with the Euler-Bernoulli theory, we can then integrate the expression for bending moment to find the equation for deflection using the double integration method.
- ✓ Macauly's Method allow us to 'turn off' partial of moment function when the value inside a bracket in that function is zero or negative.

Macaulay’s Method

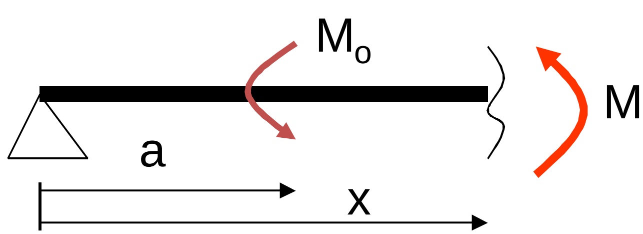
In this method, the moment function only will be considered at end of the section



$$P(x-a)^{1}$$



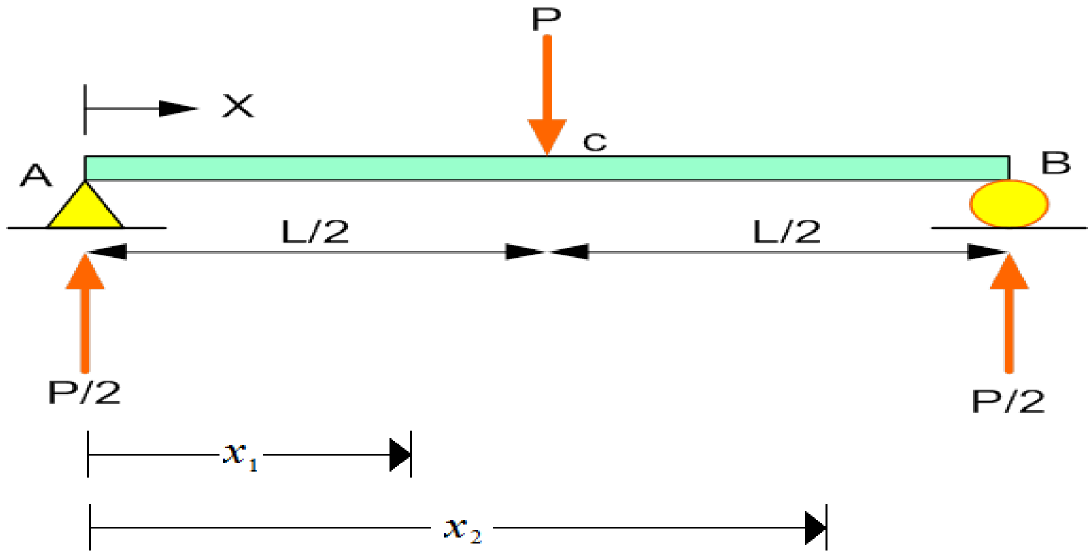
$$\frac{w(x-a)^{2}}{2}$$



$$M_{0}(x-a)^{0}$$

Macauly's Method

Let us again consider a simply supported beam AB of length L and carrying concentrated load P at mid span,C as shown below. EI is constant. This example are going to show how to find the equation of elastic curve for the beam by 'turn off' part of a function using Macauly's Method.



- Again we must write a function for the beam moment that can describe the moment for the beam wholly from the left side.
- This beam have 2 span. Macauly's Method will use the moment function to the very right with only x function as distance. Where here for example:
- Span $L/2<x<L$

$$M = \frac{P}{2}x - P\left\langle x - \frac{L}{2}\right\rangle$$

- Take note here Macauly's Method use a different bracket that have a special function that have an advanced understanding and application.

- Span L/2<x<L

$$M = \frac{P}{2}x - P\left\langle x - \frac{L}{2}\right\rangle$$

- The bracket above allow the function of 'turn off' when inside value is negative or zero.
- Means if we have $x \le \frac{L}{2}$ the $\left\langle x - \frac{L}{2}\right\rangle$ will be zero
- Mathematically explained as :

$$\left\langle x - \frac{L}{2}\right\rangle = \begin{cases} 0 \quad , & x \le \frac{L}{2} \\ x - \frac{L}{2}, & x > \frac{L}{2} \end{cases}$$

- Applying Euler-Bernoulli Theory replace M into slope and displacement integration.

$$EI\frac{d^2v}{dx^2} = \frac{P}{2}x - P\left\langle x - \frac{L}{2}\right\rangle$$

$$\theta EI = \frac{dv}{dx} = \int \frac{P}{2}x - P\left\langle x - \frac{L}{2}\right\rangle dx$$

$$vEI = f(x) = \int\int \frac{P}{2}x - P\left\langle x - \frac{L}{2}\right\rangle dx$$

- From the slope integration :

$$\theta EI = \int \frac{P}{2}x - P\left\langle x - \frac{L}{2}\right\rangle dx$$

$$\theta EI = \frac{P}{4}x^2 - \frac{P}{2}\left\langle x - \frac{L}{2}\right\rangle^2 + C_1$$

- Take note here that $\left\langle x - \frac{L}{2}\right\rangle$ is integrate as a function of x. This is rooted to advanced math that Macaulay use in his method that need to be remember.

- From the displacement integration :

$$vEI = \int\int \frac{P}{2}x - P\left\langle x - \frac{L}{2}\right\rangle dx$$

$$vEI = \int \frac{P}{4}x^2 - \frac{P}{2}\left\langle x - \frac{L}{2}\right\rangle^2 + C_1\, dx$$

$$vEI = \frac{P}{12}x^3 - \frac{P}{6}\left\langle x - \frac{L}{2}\right\rangle^3 + C_1 x + C_2$$

- Again Ttke note here that $\left\langle x - \frac{L}{2}\right\rangle$ is integrate as a function of x.

- From slope and displacement integration procedure, 2 unknown were obtained and solved using the boundary condition:
  - $v = 0, x = 0$
  - $v = 0, x = L$
- Please remember :

$$\left\langle x - \frac{L}{2} \right\rangle = \begin{cases} 0 , & x \le \frac{L}{2} \\ x - \frac{L}{2}, & x > \frac{L}{2} \end{cases}$$

- Lets use the first boundary
  - $v = 0, x = 0$

$$0EI = \frac{P}{12}0^3 - \frac{P}{6}\left\langle 0 - \frac{L}{2} \right\rangle^3 + C_1 0 + C_2$$

- Inside the bracket $\left\langle x - \frac{L}{2} \right\rangle = -\frac{L}{2} = 0$

$$0EI = \frac{P}{12}0^3 - 0 + C_1 0 + C_2$$

$$C_2 = 0$$

- The second boundary
  - $v = 0, x = L$ and $C_2 = 0$

$$0EI = \frac{P}{12}L^3 - \frac{P}{6}\left\langle L - \frac{L}{2}\right\rangle^3 + C_1 L$$

- Inside the bracket $\left\langle L - \frac{L}{2}\right\rangle = \frac{L}{2}$ we use the value

$$C_1 = -\frac{3PL^2}{48}$$

- Using unknown:

– $C_1 = -\frac{3PL^2}{48}$ and $C_2 = 0$

$$\theta EI = \frac{P}{4}x^2 - \frac{P}{2}\left\langle x - \frac{L}{2}\right\rangle^2 - \frac{3PL^2}{48}$$

$$vEI = \frac{P}{12}x^3 - \frac{P}{6}\left\langle x - \frac{L}{2}\right\rangle^3 - \frac{3PL^2}{48}x$$

- This equation can be use to obtain deflection and displacement at any position of the beam following 'turn off' rule.

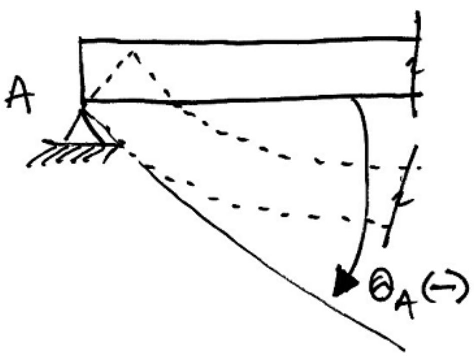
- Lets determine slope at the support :
  - At $x = 0$

$$\theta EI = \frac{P}{4}0^2 - \frac{P}{2}\left\langle 0 - \frac{L}{2}\right\rangle^2 - \frac{3PL^2}{48}$$

- Inside the bracket $\left\langle x - \frac{L}{2}\right\rangle = -\frac{L}{2} = 0$

$$\theta EI = -\frac{3PL^2}{48}$$

$$\theta = -\frac{PL^2}{16EI}$$



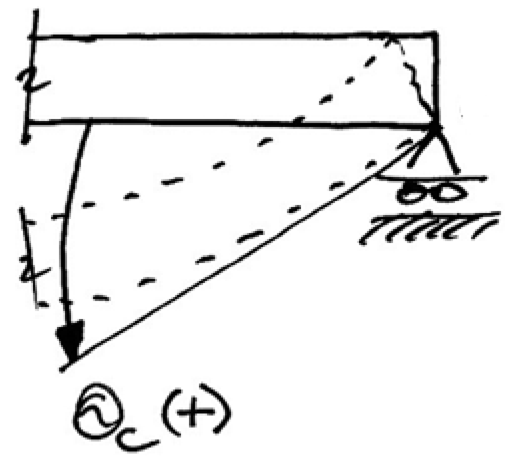
- Lets determine slope at the support :
  - At x = L

$$\theta EI = \frac{P}{4}L^2 - \frac{P}{2}\left\langle L - \frac{L}{2}\right\rangle^2 - \frac{3PL^2}{48}$$

- Inside the bracket $\left\langle x - \frac{L}{2}\right\rangle = \frac{L}{2}$

$$\theta EI = \frac{3PL^2}{48}$$

$$\theta = +\frac{PL^2}{16EI}$$



- Lets determine maximum displacement at the midspan :
  - At $x = \frac{L}{2}$

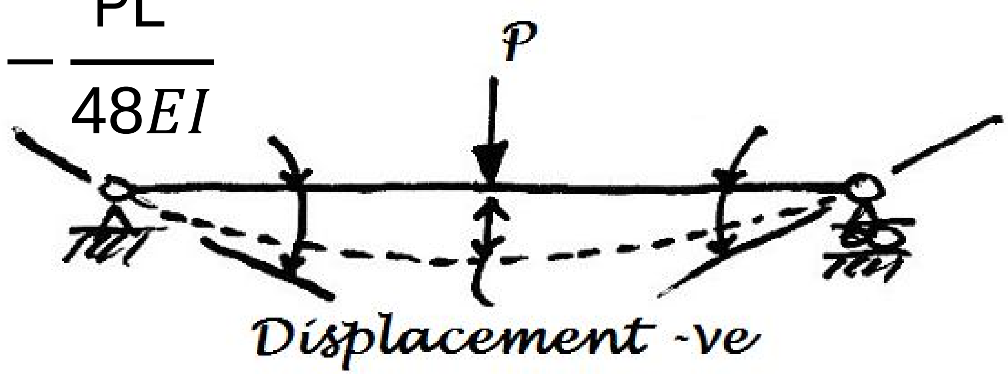
$$vEI = \frac{P}{12}\left(\frac{L}{2}\right)^3 - \frac{P}{6}\left\langle\frac{L}{2} - \frac{L}{2}\right\rangle^3 - \frac{3PL^2}{48}\left(\frac{L}{2}\right)$$

- Inside the bracket $\left\langle x - \frac{L}{2}\right\rangle = 0$

$$vEI = -\frac{PL^3}{48}$$

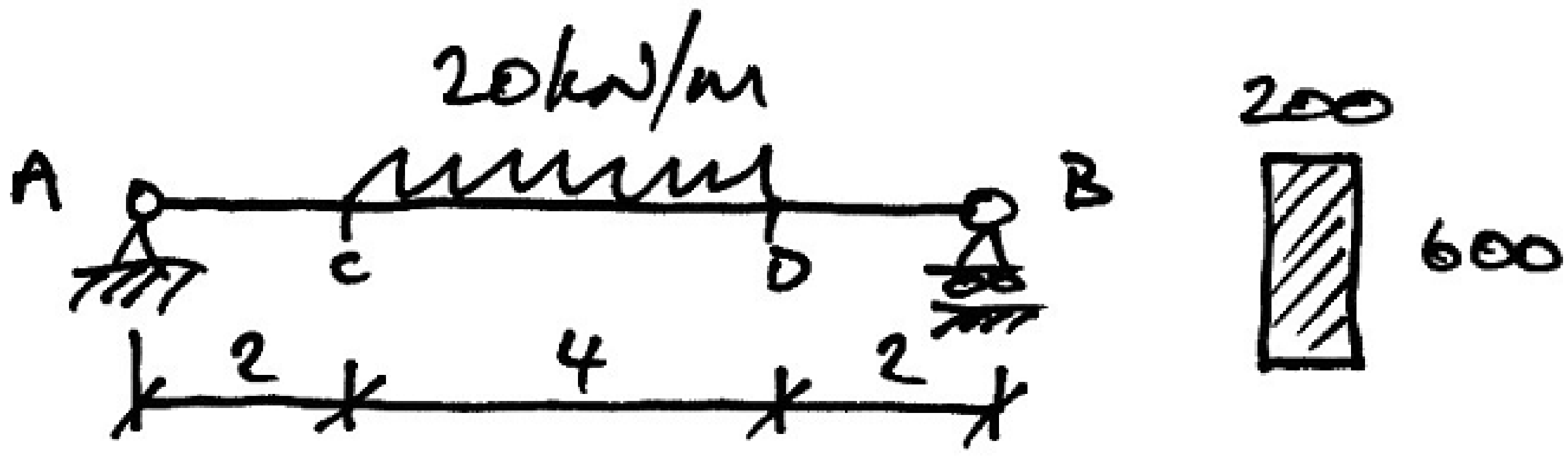
$$v = -\frac{PL^3}{48EI}$$

– Negative means downward



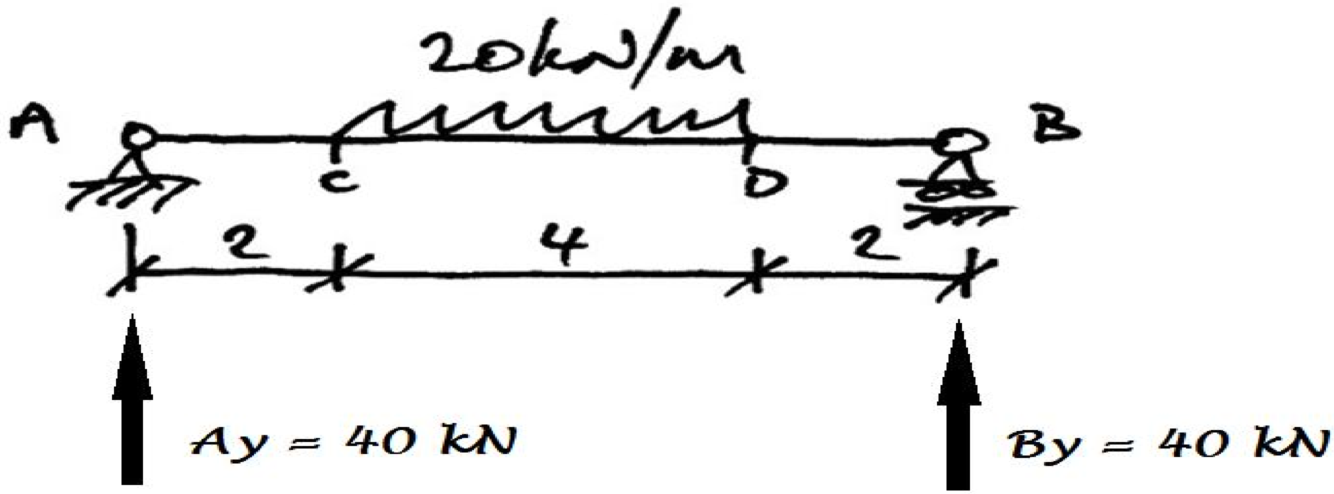
Macauly's Method

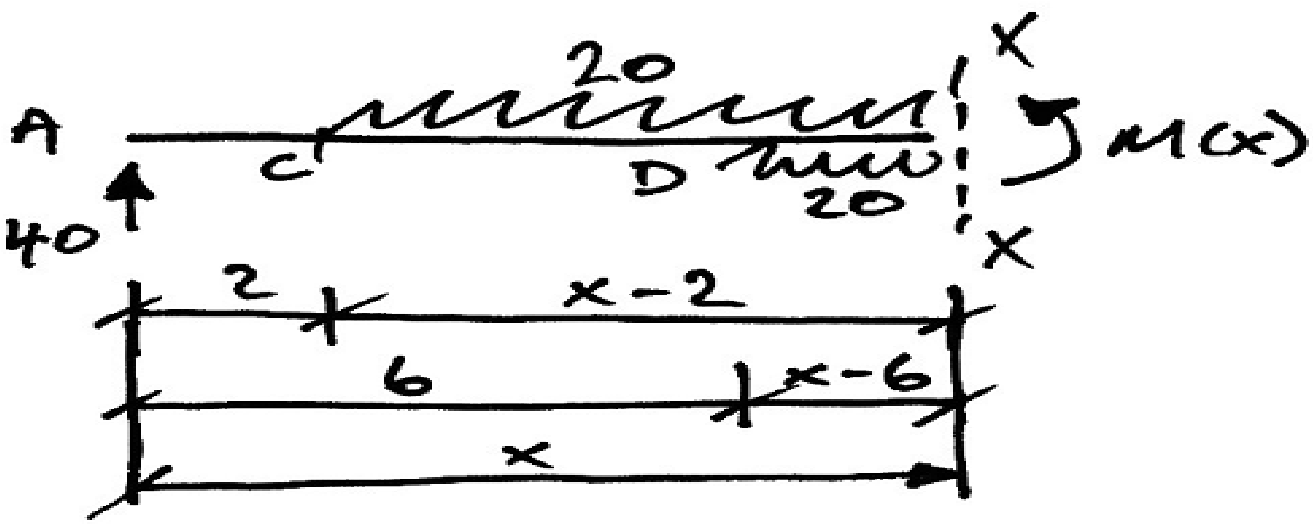
In this example we take a beam with the UDL of 20 kN/m applied to the centre of the beam as shown. The beam has the materials property, E = 30 kN/mm^2 and a cross section in mm as shown. Determine the maximum displacement in the beam

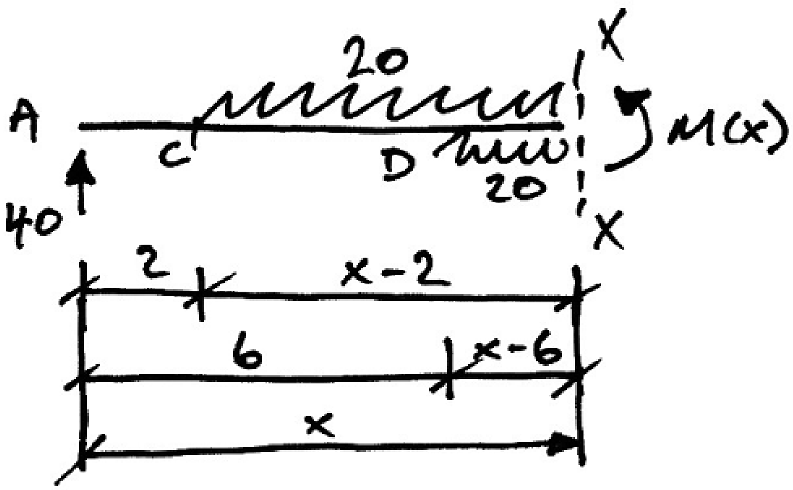


$$I = \frac{bd^3}{12} = \frac{200 \cdot 600^3}{12} = 36 \times 10^8 \text{ mm}^4$$

$$EI = \frac{(30)(36 \times 10^8)}{10^6} = 108 \times 10^3 \text{ kNm}^2$$


20kN/m
A
B
C
D
2
4
2
Ay = 40 kN
By = 40 kN


20
A
C
D
20
40
X
X
M(x)
2
x-2
6
x-6
x



- Taking moments about the cut, we have:

$$M(x) - 40x + \frac{20}{2}[x-2]^2 - \frac{20}{2}[x-6]^2 = 0$$

- Again the Macaulay brackets (take note here ⟨ ⟩ = [ ])have been used to indicate when terms should become zero. Hence:

$$M(x) = 40x - \frac{20}{2}[x-2]^2 + \frac{20}{2}[x-6]^2$$

- Applying Euler-Bernoulli (v = y):

$$M(x) = EI\frac{d^2y}{dx^2} = 40x - \frac{20}{2}[x-2]^2 + \frac{20}{2}[x-6]^2 \qquad \textbf{Equation 1}$$

- Integrate Equation 1 to get the slope equation

$$EI\frac{dy}{dx} = \frac{40}{2}x^2 - \frac{20}{6}[x-2]^3 + \frac{20}{6}[x-6]^3 + C_\theta \qquad \textbf{Equation 2}$$

- Integrate Equation 2 to get the displacement equation

$$EIy = \frac{40}{6}x^3 - \frac{20}{24}[x-2]^4 + \frac{20}{24}[x-6]^4 + C_\theta x + C_\delta \qquad \textbf{Equation 3}$$

- The boundary conditions are:
  - Support A: y =0 at x = 0
  - Support B: y =0 at x = 8

- So for the first boundary condition:

$$EI(0) = \frac{40}{6}(0)^3 - \frac{20}{24}[0-2]^4 + \frac{20}{24}[0-6]^4 + C_\theta(0) + C_\delta$$

$$C_\delta = 0$$

- For the second boundary condition:

$$EI(0) = \frac{40}{6}(8)^3 - \frac{20}{24}(6)^4 + \frac{20}{24}(2)^4 + 8C_\theta$$
$$C_\theta = -293.33$$

- Insert constants into Equations 3 (displacement)

$$EIy = \frac{40}{6}x^3 - \frac{20}{24}[x-2]^4 + \frac{20}{24}[x-6]^4 - 293.33x$$

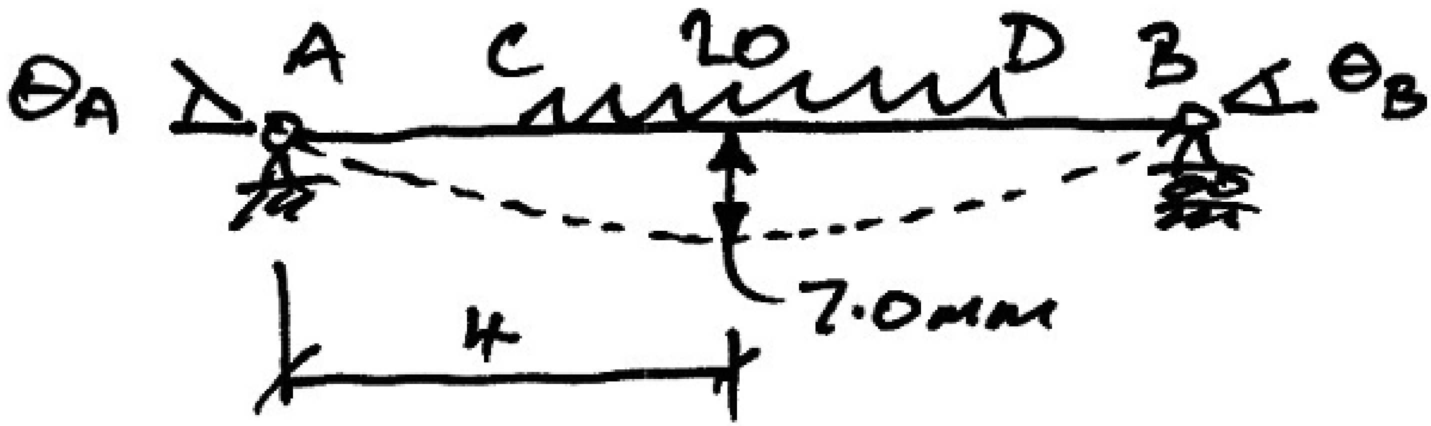
$$EI\delta_{max} = \frac{40}{6}(4)^3 - \frac{20}{24}(2)^4 + \cancel{\frac{20}{24}[4-6]^4} - 293.33(4)$$

$$= -760$$

$$\delta_{max} = \frac{-760}{EI} = \frac{-760}{108 \times 20^3} = -0.00704 \text{ m}$$

$$\delta_{max} = -7.04 \text{ mm}$$

- This is therefore a downward deflection as expected.



THANKS

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