Finite Element Analysis

Plane Truss Example

by

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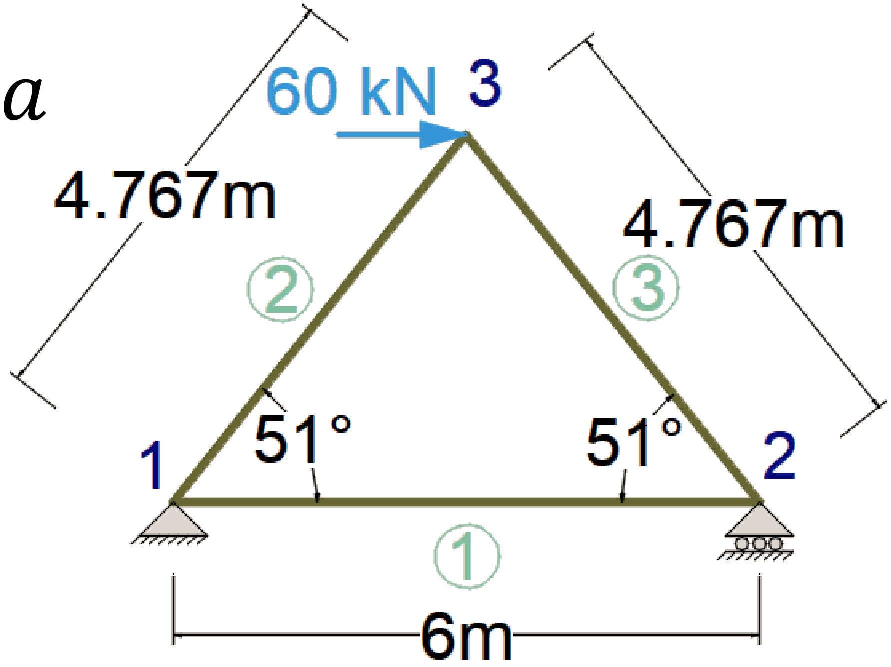
Lesson Outcomes

- At the end of this lesson, the student should be able to:
  - Apply the arbitrarily oriented bar element equations to plane truss example
  - Evaluate the plane truss using Finite Element Analysis

Plane Truss

- Analyze the plane truss shown. Relevant data is given as:
- $A = 2cm^2$
- $E = 200GPa$



Discretization

- The structure has already been discretized
- It consists of:
  - 3 nodes
  - 3 elements
  - Element 1 is connected to nodes 1 and 2, element 2 is connected to nodes 1 and 3, and element 3 is connected to nodes 2 and 3
  - Node 1 is pinned i.e. it can not move in either x or y direction
  - Node 2 is supported by a roller i.e. it can not move in the y-direction
  - 60kN force is applied on node 3
  - Element lengths are also given: Element 1 is 6m long while element 2 and element 3 are each 4.767m long

Element Stiffness Matrices

- Element stiffness matrices can be obtained by using the stiffness matrix for an arbitrarily oriented bar element developed in the previous lecture
- $[k] = \frac{AE}{L}\begin{bmatrix} C^2 & CS & -C^2 & -CS \\ CS & S^2 & -CS & -S^2 \\ -C^2 & -CS & C^2 & CS \\ -CS & -S^2 & CS & S^2 \end{bmatrix}$
- The values required for each element, therefore, are: A, E, L, C and S
- We will also tag along the relevant degrees of freedom to which an element is connected for ease in the assembly process

Stiffness Matrix for Element 1

- $\theta = 0, C = 1, S = 0$
- $A = 2cm^2 = 0.0002m^2$
- $E = 200GPa = 2 \times 10^8 kN/m^2$
- $L = 6m$
- $\frac{AE}{L} = \frac{0.0002 \times 2 \times 10^8}{6} = 6{,}666.67 kN/m$
- $$\left[k^{(1)}\right] = 6666.67 \begin{array}{c} \\ \\ \\ \\ \end{array} \begin{array}{cccc|c} u1 & v1 & u2 & v2 & \\ 1 & 0 & -1 & 0 & u1 \\ 0 & 0 & 0 & 0 & v1 \\ -1 & 0 & 1 & 0 & u2 \\ 0 & 0 & 0 & 0 & v2 \end{array}$$

Stiffness Matrix for Element 2

- $\theta = 51°, C = 0.629, S = 0.777$
- $C^2 = 0.396, S^2 = 0.604, CS = 0.489$
- $A = 2cm^2 = 0.0002m^2$
- $E = 200GPa = 2 \times 10^8 kN/m^2$
- $L = 4.767m$
- $\frac{AE}{L} = \frac{0.0002 \times 2 \times 10^8}{4.767} = 8{,}391 kN/m$
- $$[k^{(2)}] = 8391 \begin{array}{cccc|c} u1 & v1 & u3 & v3 & \\ \hline 0.396 & 0.489 & -0.396 & -0.489 & u1 \\ 0.489 & 0.604 & -0.489 & -0.604 & v1 \\ -0.396 & -0.489 & 0.396 & 0.489 & u3 \\ -0.489 & -0.604 & 0.489 & 0.604 & v3 \end{array}$$

Stiffness Matrix for Element 3

- $\theta = 129°, C = -0.629, S = 0.777$
- $C^2 = 0.396, S^2 = 0.604, CS = -0.489$
- $A = 2cm^2 = 0.0002m^2$
- $E = 200GPa = 2 \times 10^8 kN/m^2$
- $L = 4.767m$
- $\frac{AE}{L} = \frac{0.0002 \times 2 \times 10^8}{4.767} = 8{,}391 kN/m$
- $$[k^{(3)}] = 8391 \begin{array}{cccccc} & u2 & v2 & u3 & v3 & \\ \left[\begin{array}{c} \\ \\ \\ \\ \end{array}\right. & 0.396 & -0.489 & -0.396 & 0.489 & \left.\begin{array}{c} \\ \\ \\ \\ \end{array}\right] \; u2 \\ & -0.489 & 0.604 & 0.489 & -0.604 & v2 \\ & -0.396 & 0.489 & 0.396 & -0.489 & u3 \\ & 0.489 & -0.604 & -0.489 & 0.604 & v3 \end{array}$$

Assembly of Structure Stiffness Matrix

- Using direct stiffness assembly:

- $[K] =$

$$\begin{bmatrix} 9989.51 & 4103.2 & -6666.67 & 0 & -3322.84 & -4103.2 \\ 4103.2 & 5068.16 & 0 & 0 & -4103.2 & -5068.16 \\ -6666.67 & 0 & 9989.51 & -4103.2 & -3322.84 & 4103.2 \\ 0 & 0 & -4103.2 & 5068.16 & 4103.2 & -3322.84 \\ -3322.84 & -4103.2 & -3322.84 & 4103.2 & 6645.68 & 0 \\ -4103.2 & -5068.16 & 4103.2 & -3322.84 & 0 & 10136.32 \end{bmatrix}$$

System of Equations

- $$\begin{Bmatrix} f_{1x} \\ f_{1y} \\ f_{2x} \\ f_{2y} \\ f_{3x} \\ f_{3y} \end{Bmatrix} = \begin{bmatrix} 9989.51 & 4103.2 & -6666.67 & 0 & -3322.84 & -4103.2 \\ 4103.2 & 5068.16 & 0 & 0 & -4103.2 & -5068.16 \\ -6666.67 & 0 & 9989.51 & -4103.2 & -3322.84 & 4103.2 \\ 0 & 0 & -4103.2 & 5068.16 & 4103.2 & -5068.16 \\ -3322.84 & -4103.2 & -3322.84 & 4103.2 & 6645.68 & 0 \\ -4103.2 & -5068.16 & 4103.2 & -5068.16 & 0 & 10136.32 \end{bmatrix} \begin{Bmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \\ u_3 \\ v_3 \end{Bmatrix}$$

Boundary Conditions

- We know that:
- $u_1 = v_1 = v_1 = 0$
- $f_{2x} = f_{3y} = 0$, and $f_{3x} = 60kN$
- These boundary conditions can be applied by removing the 1st, 2nd, and 4th rows and columns from the system of equations and inserting the relevant values in the force vector

Reduced System of Equations

- The reduced system of equations is given as:
- $\begin{Bmatrix} 0 \\ 60 \\ 0 \end{Bmatrix} = \begin{bmatrix} 9989.51 & -3322.84 & 4103.2 \\ -3322.84 & 6645.68 & 0 \\ 4103.2 & 0 & 10136.32 \end{bmatrix} \begin{Bmatrix} u_2 \\ u_3 \\ v_3 \end{Bmatrix}$
- This system of equations can be solved using any method applicable to such systems

Solution

- From the solution of the system of equations, we get:
- $u_2 = 0.0045m = 4.5mm$
- $u_3 = 0.011278m = 11.28mm$
- $v_3 = -0.00182 = -1.82mm$
- These values show that both nodes 2 and 3 are moving towards the right by 4.5mm and 11.28mm, respectively (negative values would have suggested leftwards movement)
- Node 3 is also moving 1.82mm downwards (a positive value would have suggested upwards movement)

Support Reactions

- The unknown support reactions can be obtained by inserting the calculated deformations into the equations that we removed earlier
- $f_{1x} = -6666.67u_2 - 3322.84u_3 - 4103.2v_3 = -60.0045 \cong -60kN$
- $f_{1y} = -4103.2u_3 - 5068.16v_3 = -37.0454$
- $f_{2y} = -4103.2u_2 + 4103.2u_3 - 5068.16v_3 = 37.0454$
- We can verify these results by applying simple equilibrium to the structure

Verification through Equilibrium

- $\sum M@1 = 0$
- $60 \times 3.7047 - f_{2y} \times 6 = 0$
- $f_{2y} = 37.047kN$
- $\sum F_y = 0$
- $f_{1y} + 37.047 = 0$
- $f_{1y} = -37.047kN$
- $\sum F_x = 0$
- $f_{1x} + 60 = 0$
- $f_{1x} = -60kN$
- We can see that the values obtained from FEA are, within limit, equal to those obtained by simple equilibrium equations

Author Information

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