Mathematics for Management

Chapter 1: Functions

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Content:

- 1.0 Introduction
- 1.1 Functions
- 1.2 Special Functions
- 1.3 Combination of Functions
- 1.4 Composition of Functions
- 1.5 Inverse Functions
Expected Outcome:

Upon the completion of this course, students will have the ability to:

1. Obtain the domain, function values, equality of function and difference quotient of a function.
2. Identify the types of special functions i.e. constant functions, polynomial functions, rational functions, case-defined function, and absolute value function.
3. Find the solution of combination of functions, composition of functions and inverse functions.
1.0 Introduction

Definition:

- A **function** is a relationship in which each input number is paired with **exactly one** output number.

- The set of all input numbers is called the **domain** of the function (independent variable)

- The set of all output numbers is called the **range** (dependent variable)
1.1 Functions

\[ y = f(x) \]

Output | Name of Function | Input
---|---|---

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Domain

Domain is the set of all input numbers

Three different cases of domain:

1. \[ f(x) = \frac{x}{x^2 + x - 6} \]
   
   \[ x^2 + x - 6 \neq 0 \]
   
   \[ (x - 2)(x + 3) \neq 0 \]
   
   \[ x \neq 2, -3 \]

   Domain: All real numbers except 2 and -1

Fractional functions
(The denominator should not be equal to zero)
Domain

2

\[ g(k) = \sqrt{2k - 5} \]

\[ 2k - 5 \geq 0 \]

\[ 2t \geq 5 \]

\[ t \geq \frac{5}{2} \]

Domain: \( \left[ \frac{5}{2}, \infty \right) \)

Square root functions
(Under the square root, all numbers should be in positive)

3

\[ g(x) = 6x^2 - x + 7 \]

Domain: All real numbers

Polynomial functions
(all numbers can be as an input)
Function Values

Function values is the output of the function corresponding to the input

Example 1.1

Let \( g(p) = 3p^2 - p + 5 \) find \( g(x), g(c^2), g(k + h) \)

Solution:

\[
g(z) = 3z^2 - z + 5
\]

\[
g(r^2) = 3(r^2)^2 - r^2 + 5 = 3r^4 - r^2 + 5
\]

\[
g(x + h) = 3(x + h)^2 - (x + h) + 5
\]

\[
= 3(x^2 + 2hx + h^2) - x - h + 5
\]

\[
= 3x^2 + 6hx + 3h^2 - x - h + 5
\]
Exercises

Function values is the output of the function corresponding to the input

Example 1.1

Let \( g(z) = 3z^2 - z + 5 \) find \( g(x), g(h^2), g(k + h) \)

Solution: \( g(x) = 3x^2 - x + 5 \)

\[
\begin{align*}
g(h^2) &= 3(h^2)^2 - h^2 + 5 = 3h^4 - h^2 + 5 \\
g(k + h) &= 3(k + h)^2 - (k + h) + 5 \\
&= 3(k^2 + 2hk + h^2) - k - h + 5 \\
&= 3k^2 + 6hk + 3h^2 - k - h + 5
\end{align*}
\]
Find the **domain** and the **function values** for the following functions:

(a) \( f(x) = \sqrt{x + 8}; \ f(3), f(8), f(-4) \)
(b) \( g(x) = x^3 + x^2; \ g(2), g(-1), g(t) \)
(c) \( h(x) = \frac{1}{x - 6} \)
Equality of Functions

To say that two functions \( f \) and \( g \) are equal, denoted \( f = g \), is to say that

- the domain of \( f \) = domain of \( g \)
- for every \( x \) in the domain of \( f \) and \( g \), output \( f(x) = g(x) \)
Example:

Determine whether the following functions are equal.

(a) \( f(x) = (x + 1)^2 \) and \( g(x) = x^2 + 2x + 1 \)

(b) \( f(x) = x^2 \) and \( g(x) = x^2 \) for \( x \geq 0 \)
SOLUTION:

(a) Domain $f(x) = \text{all real numbers.}$
Domain $g(x) = \text{all real numbers.}$

Therefore, Domain $f(x) = \text{Domain } g(x)$

\[ f(x) = (x+1)^2 \]
\[ = x^2 + 2x + 1 \]

\[ g(x) = x^2 + 2x + 1 \]

\[ \therefore f(x) = g(x) \]

Therefore, these two functions are equal.

(b) If we have $f(x) = x^2$, with no explicit mention of domain, and $g(x) = x^2$ for $x \geq 0$, then domain $f(x) \neq g(x)$. Here the domain of $f$ is all real numbers and the domain for $g$ is $[0, \infty)$. Therefore, these two functions are not equal.
Exercises:

Determine whether the following functions are equal.

(a) \( f(x) = (x+1)^2 \) and \( g(x) = x + 2 \)

(b) \( g(x) = x + 2 \) and \( k(x) = \begin{cases} x + 2 & \text{if } x \neq 1 \\ 3 & \text{if } x = 1 \end{cases} \)
DIFFERENT QUOTIENT

The difference quotient of a function is an important mathematical concept. The expression is

$$\frac{f(x+h) - f(x)}{h}$$

Example: If $f(x) = x^2$, find $\frac{f(x+h) - f(x)}{h}$

$$\frac{f(x+h) - f(x)}{h} = \frac{(x+h)^2 - x^2}{h}$$

$$= \frac{x^2 + 2hx + h^2 - x^2}{h}$$

$$= \frac{2hx + h^2}{h}$$

$$= \frac{h(2x + h)}{h}$$

$$= 2x + h$$
Different Quotient

The difference quotient of a function is an important mathematical concept. The expression is

\[ \frac{f(x + h) - f(x)}{h} \]

**Example:** If \( f(x) = x^2 \) find

\[ \frac{f(x + h) - f(x)}{h} = \frac{(x + h)^2 - x^2}{h} \]

\[ = \frac{x^2 + 2hx + h^2 - x^2}{h} \]

\[ = \frac{2hx + h^2}{h} = \frac{h(2x + h)}{h} = 2x + h \]
Exercises:

Find the **difference quotient** of the following functions.

(a) Find \( \frac{-f(x+h)+f(x)}{h} \) if \( f(x) = \frac{x}{2} + 1 \)

(b) If \( f(x) = 4x - 5 \), find \( \frac{f(x+2h) - f(x)}{4h} \).
Special Functions

• Constant Functions
• Polynomial Functions
• Rational Functions
• Case-Defined Function
• Absolute-Value Function
Constant Functions

- A function of the form $h(x) = c$, where $c$ is a constant, is called a constant function.

Let $h(x) = 7$, so

- $h(12) = 7$,
- $h(-385) = 7$,
- $h(x + 5) = 7$

We call $h$ a constant function because all the function value are the same.
In general, a function of the form

\[ f(x) = c_n x^n + c_{n-1} x^{n-1} + \ldots + c_1 x + c_0 \]

Where \( n \) is a nonnegative integer and \( c_n, c_{n-1}, \ldots, c_0 \) are constant with \( c_n \neq 0 \), is called a polynomial function.
Example:

(a) \( f(x) = x^3 - 6x^2 + 7 \) → is a polynomial function of degree 3 with leading coefficient 1.

(b) \( g(x) = \frac{2x}{3} \) → is a linear function with leading coefficient \( \frac{2}{3} \).

(c) \( f(x) = \frac{2}{x^3} \) → is NOT a polynomial function. Because \( f(x) = x^{-3} \) and the exponent for \( x \) is not a nonnegative integer, this function does not have the proper form for a polynomial.
A function that is a quotient of polynomial functions is called a rational function.

\( f(x) = \frac{x^2 - 6x}{x + 5} \rightarrow \) is a rational function, since the numerator and denominator are each polynomials. Note that this rational function is not defined for \( x = 5 \).

\( g(x) = 2x + 3 \rightarrow \) is a rational function, since \( 2x + 3 = \frac{2x + 3}{1} \). In fact, every polynomial function is also a rational function.
Case-Defined Function

Let

\[ F(s) = \begin{cases} 
1 & \text{if } -3 \leq s \leq 2 \\
0 & \text{if } 2 \leq s \leq 3 \\
s - 3 & \text{if } 3 < s \leq 8 
\end{cases} \]

This is called a case-defined function because the rule for specifying it is given by rules for each of several disjoint cases.
Absolute-Value Function

The function \( |x| \) is called the absolute-value function. \( |x| \) is defined by

\[
|x| = \begin{cases} 
  x & \text{if } x \geq 0 \\
  -x & \text{if } x < 0 
\end{cases}
\]
Classify each of the special function. Then, find the function values.

(a) \( f(x) = 8; f(2), f(t+8), f(-\sqrt{17}) \)

(b) \( H(x) = \frac{1}{\pi} - 3x^5 + 2x^6 + x^7; H(2), H(-2), H(1) \)

(c) \( h(x) = \frac{x^2 + x}{x^3 + 4}; h(2), h(t+8), h(0) \)

(d) \( G(\theta) = \begin{cases} 2\theta - 5 & \text{if } \theta \leq 2 \\ \theta^2 - 3\theta + 1 & \text{if } \theta > 2 \end{cases} \)

\( G(3), G(-3), G(2) \)

(e) \( g(x) = |x - 3|; g(10), g(3), g(-3) \)
Combinations of Functions

• There are several ways of combining two functions to create a new function. For example, we can combine the functions by addition, subtraction, multiplication, division, multiplication by a constant, and composition.

• Suppose $f$ and $g$ are the functions given by

\[ f(x) = x^3 \quad \text{and} \quad g(x) = 5x \]

Adding $f(x)$ and $g(x)$ gives

\[ f(x) + g(x) = x^3 + 5x \]
• In general, for any function \( f \) and \( g \), we define the sum \( f + g \), the difference \( f - g \), the product \( fg \) and the quotient \( f/g \) and scalar product \( cf(x) \) as follows:

\[
\begin{align*}
  \text{i.} & \quad (f + g)(x) = f(x) + g(x) \\
  \text{ii.} & \quad (f - g)(x) = f(x) - g(x) \\
  \text{iii.} & \quad (fg)(x) = f(x) \cdot g(x) \\
  \text{iv.} & \quad \frac{f}{g}(x) = \frac{f(x)}{g(x)}, \quad g(x) \neq 0 \\
  \text{v.} & \quad (cf)(x) = c \cdot f(x)
\end{align*}
\]
Example:

If \( f(x) = x^2 \) and \( g(x) = 3x \), therefore we have

Solution:

(a) \((f + g)(x) = f(x) + g(x) = x^2 + 3x\)
(b) \((f - g)(x) = f(x) - g(x) = x^2 - 3x\)
(c) \((fg)(x) = f(x) \cdot g(x) = x^2(3x) = 3x^3\)
(d) \(\frac{f}{g}(x) = \frac{f(x)}{g(x)} = \frac{x^2}{3x} = \frac{x}{3} \quad \text{for} \quad x \neq 0\)
(e) \((\sqrt{2}f)(x) = \sqrt{2}f(x) = \sqrt{2}x^2\)
Exercises:

If \( f(x) = 3x - 1 \) and \( g(x) = x^2 + 3x \), find

(a) \( (f + g)(3) \)
(b) \( (f - g)(\frac{1}{2}) \)
(c) \( (fg)(-\frac{1}{2}) \)
(d) \( \frac{f}{g}(-2) \)
(e) \( (3f)(-\sqrt{2}) \)
Composition of Functions

**Definition:**

If $f$ and $g$ are functions, the composite of $f$ with $g$ is the function $f \circ g$ defined by

$$(f \circ g)(x) = f(g(x))$$

where the domain of $f \circ g$ is the set of all those $x$ in the domain of $g$ such that $g(x)$ is in the domain of $f$. 
Example:

If \( f(x) = \sqrt{x} \) and \( g(x) = x + 1 \), find

(a) \((f \circ g)(x)\), then \((f \circ g)(8)\).

(b) \((g \circ f)(x)\), then \((g \circ f)\left(\frac{1}{4}\right)\).
Solution:

(a) \((f \circ g)(x) = f(g(x))\)
\[= f(x+1)\]
\[= \sqrt{x+1}\]

The domain is all \(x \geq -1\), or equivalently, the interval \([-1, \infty)\).

When \(x = 8\), then \((f \circ g)(8) = \sqrt{8+1} = \sqrt{9} = 3\).

(b) \((g \circ f)(x) = g(f(x))\)
\[= g(\sqrt{x})\]
\[= \sqrt{x} + 1\]

The domain is all \(x \geq 0\), or equivalently, the interval \([0, \infty)\).

When \(x = \frac{1}{4}\), then \((g \circ f)\left(\frac{1}{4}\right) = \sqrt{\frac{1}{4}} + 1 = \frac{1}{2} + 1 = \frac{3}{2}\).
Exercises:

If \( f(x) = 2x + 1 \) and \( g(x) = \frac{3}{2x + 1} \), find

(a) \( (g \circ f)(-2) \)

(b) \( (f \circ g)\left(\frac{1}{2}\right) \)

.
Inverse Functions

• Only exist for one-to-one function

\[ f(x) = y \Rightarrow f^{-1}(y) = x \]

• Remarks: To verify that is \( f^{-1} \) the inverse of \( f \), show that

\[ f^{-1}[f(x)] = f[f^{-1}(x)] = x \]
A function $f$ that satisfies

for all $a$ and $b$, if $f(a) = f(b)$, then $a = b$

is called **one-to-one** function.

3 steps to find the inverse of a function, $f$:

**STEP 1:** Replace $f(x)$ with $y$

**STEP 2:** Solve for $x$ in term of $y$ obtaining $x = g(y)$

**STEP 3:** Replace $x$ with $f^{-1}(x)$. Then, $f^{-1}(x) = g(x)$
Example:

If \( f(x) = (x - 1)^2 \) for \( x \geq 1 \), find \( f^{-1}(x) \).

Solution:

Let \( y = (x - 1)^2 \), for \( x \geq 1 \).
Then, \( x - 1 = \sqrt{y} \) and hence \( x = \sqrt{y} + 1 \).
Therefore, we have \( f^{-1}(x) = \sqrt{x} + 1 \).

Check:

\[
\begin{align*}
    f^{-1}(f(x)) &= f\left((x - 1)^2\right) = \sqrt{(x - 1)^2} + 1 \\
                   &= x - 1 + 1 = x
\end{align*}
\]

\[
\begin{align*}
    f(f^{-1}(x)) &= f\left(\sqrt{x} + 1\right) = \left(\sqrt{x} + 1\right)^2 \\
                   &= (\sqrt{x} + 1)^2 - 2(\sqrt{x} + 1) + 1 = x
\end{align*}
\]

Therefore, the inverse of function \( f \) given above is \( f^{-1}(x) = \sqrt{x} + 1 \).
Exercises:

Find the inverse of the given function.

(a) \( f(x) = \frac{3x-1}{x+5} \)

(b) \( g(x) = (2x+8)^3 \)
THE END

~THANK YOU~
Author Information

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