AIMS

This chapter is aimed to compute the root(s) of the equations by using graphical method and numerical methods.

EXPECTED OUTCOMES

1. Students should be able to find roots of the equations by using graphical approach and incremental search.
2. Students should be able to find the roots of the equations by using bracketing and open methods.
3. Students should be able to provide the comparison between bracketing and open methods.
4. Students should be able to calculate the approximate and true percent relative error.

REFERENCES

Content

1. Introduction
2. Graphical Method
3. Incremental Search
4. Bracketing Method
   - Bisection Method
   - False-Position Method
5. Open Method
   - Newton Raphson Method
   - Secant Method
INTRODUCTION

- Mathematical model in science and engineering involve equations that need to be solved.

- Equation of one variable can be formulated as

\[ f(x) = 0 \]  \hspace{1cm} (1)

- Equation (1) can be in the form of linear and nonlinear.

- Solving equation (1) means that finding the values of \( x \) that satisfying equation (1).
Equation (1) may belong to one of the following types of equations:

1. Algebraic equations
2. Polynomial equations
3. Transcendental equations

A non-algebraic equation of trigonometric, exponential and logarithm function
Example 1: Algebraic Equation

\[ 4x - 3x^2y - 15 = 0 \]

Example 2: Polynomial Equation

\[ x^2 + 2x - 4 = 0 \]

Example 3: Transcendental Equation

\[ \sin(2x) - 3x = 0 \]
Finding Roots for Quadratic Equations

\[ f(x) = ax^2 + bx + c \]

**INTRODUCTION (Cont.)**

Analytical Methods

1. Factorization
2. Completing the Square
3. Quadratic Formula

Numerical Methods by Norhayati Rosli
INTRODUCTION (Cont.)

- All above mentioned methods to solve quadratic equations are analytical methods.
- The solution obtained by using analytical methods is called exact solution.
- Due to the complexity of the equations in modelling the real life system, the exact solutions are often difficult to be found.
- Thus require the used of numerical methods.
- The solution that obtained by using numerical methods is called numerical solution.
INTRODUCTION (Cont.)

Three types of Numerical Methods shall be considered to find the roots of the equations:

1. Incremental Search

2. Bracketing Methods
   - Bisection Method
   - False Position Method

3. Open Methods
   - Newton Raphson Method
   - Secant Method

Prior to the numerical methods, a **graphical method** of finding roots of the equations are presented.
GRAPHICAL METHOD

- Graphical method is the simplest method.
- The given function is plotted on Cartesian coordinate and $x$ –values (roots) that satisfying $f(x) = 0$ is identified.
- $x$ –values (roots) satisfying $f(x) = 0$ provide approximation roots for the underlying equations.
- $f(x)$ can have one or possibly many root(s).
GRAPHICAL METHOD (Cont.)

Figure 1: One Solution

Figure 2: Two Solutions
GRAPHICAL METHOD (Cont.)

Figure 3 : Many Solutions
**Example 4**

Find root(s) of \( f(x) = x^2 - 8x + 3 \) by using graphical method.

**Solution**

Based on the graph, the function \( f(x) \) cross \( x \) -axis at two points.
Therefore there are two roots for \( f(x) \)
The approximate roots of \( f(x) \) are 0.364 and 7.663
Example 5

Find root(s) of \( f(x) = \cos(x) + \sin(3x) \) for \( 0 \leq x \leq 4\pi \) by using graphical method.

Solution

There are twelve roots for \( f(x) \) since the function cross \( x \)-axis at twelve points. The approximate roots of \( f(x) \) are 1.238, 2.401, 2.701, 4.239, 5.439, 5.852, 7.39, 8.628, 8.966, 10.691, 11.704 and 12.154.
Example 6 [Chapra & Canale]

The velocity of a free falling parachutist is given as

\[ v = \frac{gm}{c} \left( 1 - e^{-\frac{c}{m}t} \right) \]

Use the graphical approach to determine the drag coefficient, \( c \) needed for a parachutist of mass, \( m = 68.1 \) kg to have a velocity of \( 40 \, \text{ms}^{-1} \) after free falling for time, 10s. Given also gravity is \( 9.8 \, \text{ms}^{-2} \)

Solution

To determine the root of drag coefficient, \( c \). we need to have a function \( f(c) = 0 \).

Substituting the values given in the equation and rearranging yield

\[ f(c) = \frac{9.8(68.1)}{c} \left( 1 - e^{-\frac{c}{68.1}10} \right) - 40 = 0 \]
Graphical Method (Cont.)

Solution (cont.)

Plot the function $f(c)$ and determine where the graph crosses the horizontal axis.

<table>
<thead>
<tr>
<th>$x$</th>
<th>$f(x)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>4</td>
<td>34.115</td>
</tr>
<tr>
<td>8</td>
<td>17.653</td>
</tr>
<tr>
<td>12</td>
<td>6.0670</td>
</tr>
<tr>
<td>16</td>
<td>-2.2690</td>
</tr>
<tr>
<td>20</td>
<td>-8.4010</td>
</tr>
</tbody>
</table>

From the graphical view, the root exists between $c = 12$ and $c = 16$, where the functions $f(12)$ and $f(16)$ have opposite sign, that is $f(12) \times f(16) < 0$. 

Numerical Methods
by Norhayati Rosli
INCREMENTAL SEARCH

- Incremental search is a technique of calculating $f(x)$ for incremental values of $x$ over the interval where the root lies.
- It starts with an initial value, $x_0$.
- The next value $x_n$ for $n = 1, 2, 3, ...$ is calculated by using
  
  $$x_n = x_{n-1} + h$$
  
  where $h$ is referred to a step size.
- If the sign of two $f(x)$ changes or if
  
  $$f(x_n) \cdot f(x_{n-1}) < 0$$
  
  then the root exist over the prescribed interval of the lower bound, $x_l$ and upper bound, $x_u$.
- The root is estimated by using
  
  $$x_r = \frac{x_l + x_u}{2}$$
INCREMENTAL SEARCH (Cont.)

Example 6

Find the first root of \( f(x) = 4.15x^2 - 16x + 8 \) by using incremental search. Start the procedure with the initial value, \( x_0 = 0 \) and the step size, \( h = 0.1 \). Perform three iterations of the incremental search to achieve the best approximation root.

Solution

Start the estimation with initial value \( x_0 = 0 \) and step size, \( h = 0.1 \).

<table>
<thead>
<tr>
<th>( x )</th>
<th>( f(x) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>8</td>
</tr>
<tr>
<td>0.1</td>
<td>6.4415</td>
</tr>
<tr>
<td>0.2</td>
<td>4.966</td>
</tr>
<tr>
<td>0.3</td>
<td>3.5735</td>
</tr>
<tr>
<td>0.4</td>
<td>2.264</td>
</tr>
<tr>
<td>0.5</td>
<td>1.0375</td>
</tr>
<tr>
<td>0.6</td>
<td>-0.106</td>
</tr>
</tbody>
</table>

\[
f(0.5) \cdot f(0.6) < 0
\]

\[
x_r = \frac{0.5 + 0.6}{2} = 0.55
\]

Numerical Methods
by Norhayati Rosli
INCREMEMNTAL SEARCH (Cont.)

Solution (Cont.)

Increasing the accuracy of root estimation with step size, \( h = 0.01 \)
for \( x \in [0.5, 0.6] \)

<table>
<thead>
<tr>
<th>( x )</th>
<th>( f(x) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.5</td>
<td>1.0375</td>
</tr>
<tr>
<td>0.51</td>
<td>0.919415</td>
</tr>
<tr>
<td>0.52</td>
<td>0.80216</td>
</tr>
<tr>
<td>0.53</td>
<td>0.685735</td>
</tr>
<tr>
<td>0.54</td>
<td>0.57014</td>
</tr>
<tr>
<td>0.56</td>
<td>0.455375</td>
</tr>
<tr>
<td>0.57</td>
<td>0.34144</td>
</tr>
<tr>
<td>0.58</td>
<td>0.11606</td>
</tr>
<tr>
<td>0.59</td>
<td>0.004615</td>
</tr>
<tr>
<td>0.60</td>
<td>-0.106</td>
</tr>
</tbody>
</table>

\[
f(0.59) \cdot f(0.6) < 0
\]

\[
x_r = \frac{0.59 + 0.6}{2} = 0.5950
\]
Increasing the accuracy of root estimation with step size, \( h = 0.001 \) for \( x \in [0.59, 0.6] \)

<table>
<thead>
<tr>
<th>( x )</th>
<th>( f(x) )</th>
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</thead>
<tbody>
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<td>0.59</td>
<td>0.004615</td>
</tr>
<tr>
<td>0.591</td>
<td>-0.0064385</td>
</tr>
<tr>
<td>0.592</td>
<td>-0.0175744</td>
</tr>
<tr>
<td>0.593</td>
<td>-0.02865665</td>
</tr>
<tr>
<td>0.594</td>
<td>-0.097306</td>
</tr>
</tbody>
</table>

\[
\begin{align*}
\text{For three iterations, the first root of } f(x) &= 4.15x^2 - 16x + 8 \text{ is 0.5905 with } \epsilon_a = 0.76\% \\
\end{align*}
\]
Figure 1 illustrates the basic idea of bracketing method—that is guessing an interval containing the root(s) of a function.

Starting point of the interval is a lower bound, $x_l$. End point of the interval is an upper bound, $x_u$.

By using bracketing methods, the interval will split into two subintervals and the size of the interval is successively reduced to a smaller interval.

The subintervals will reduce the range of intervals until its distance is less than the desired accuracy of the solution.

Figure 4: Graphical Illustration of Bracketing Method
Bracketing methods always converge to the true solution.
There are two types bracketing methods; bisection method and false position method.
Bisection method is the simplest bracketing method. The lower value, $x_l$ and the upper value, $x_u$ which bracket the root(s) are required.

The procedure starts by finding the interval $[x_l, x_u]$ where the solution exist.

As shown in Figure 5, at least one root exist in the interval $[x_l, x_u]$ if $f(x_l) \cdot f(x_u) < 0$.

**Figure 5: Solution of $f(x) = 0$**
BISECTION METHOD (Cont.)

For the continuous equation of one variable, \( f(x) = 0 \),

**Step 1:** Choose the lower guess, \( x_l \) and the upper guess, \( x_u \) that bracket the root such that the function has opposite sign over the interval, \( x_l \leq x \leq x_u \).

**Step 2:** The estimation root, \( x_r \) is computed by using

\[
x_r = \frac{x_l + x_u}{2}
\]

**Step 3:** Use the following evaluations to identify the subinterval that the root lies
- If \( f(x_l) \cdot f(x_r) < 0 \), then the root lies in the lower subinterval. Therefore, set \( x_u = x_r \) and repeat **Step 2**.
- If \( f(x_l) \cdot f(x_r) > 0 \), then the root lies in the upper subinterval. Therefore set \( x_l = x_r \) and repeat **Step 2**.
- If \( f(x_l) \cdot f(x_r) = 0 \), then the root is equal to \( x_r \). Terminate the computation.

**Step 4:** Calculate the approximate percent relative error,

\[
\varepsilon_a = \left| \frac{x_r^{\text{present}} - x_r^{\text{previous}}}{x_r^{\text{present}}} \right| \times 100\%
\]

**Step 5:** Compare with. If \( \varepsilon_a < \varepsilon_s \), then stop the computation. Otherwise go to **Step 2** and repeat the process by using the new interval.
Example 7

Use three iterations of the bisection method to determine the root of \( f(x) = -0.6x^2 + 2.4x + 5.5 \). Employ initial guesses, \( x_l = 5 \) and \( x_u = 10 \). Compute the approximate percent relative error, \( \varepsilon_a \) and true percent relative error, \( \varepsilon_t \) after each iteration.

Solution

Calculate the true value for the given quadratic function \( f(x) = -0.6x^2 + 2.4x + 5.5 \) using quadratic formula (or you can calculate directly by using the calculator)

\[
x = \frac{-2.4 \pm \sqrt{(2.4)^2 - 4(-0.6)(5.5)}}{2(-0.6)}
\]

\[
x = -1.6286, \quad x = 5.6286
\]

Choose the true value, \( x = 5.6286 \) for the highest root of \( f(x) \). Estimate the root of \( f(x) \) using bisection method with initial guess \( x_l = 5 \) and \( x_u = 10 \).
BISECTION METHOD (Cont.)

Solution (Cont.)

Estimate the root of $f(x)$ using bisection method with initial interval $[5,10]$.

First iteration, $x \in [5,10]$

\[
f(5) = 2.50 \\
f(10) = -30.50
\]

First estimate using bisection method formula

\[
x_r = \frac{5 + 10}{2} = 7.5
\]

\[
f(7.5) = -10.25
\]

Since $f(x_l) \cdot f(x_r) < 0$, the root lies in the lower subinterval. Then set $x_u = 7.5$.

\[
\varepsilon_t = \left| \frac{5.6286 - 7.5}{5.6286} \right| \times 100\% = 33.23\% \text{ and } \varepsilon_a = -
\]
Solution (Cont.)

Second iteration, \( x \in [7.5, 10] \)

\[
f(5) = -10.25
\]

\[
f(10) = -30.50
\]

First estimate using bisection method formula

\[
x_r = \frac{7.5 + 10}{2} = 6.25
\]

\[
f(6.25) = -2.9375
\]

Since \( f(x_l) \cdot f(x_r) < 0 \), the root lies in the lower subinterval. Then set \( x_u = 6.25 \).

\[
\varepsilon_t = \left| \frac{5.6286 - 6.25}{5.6286} \right| \times 100\% = 11.04\% \text{ and } \varepsilon_a = 20\%
\]
BISECTION METHOD (Cont.)

Solution (Cont.)

Continue the third iteration for \( x \in [5, 6.25] \). The results are summarized in the following table.

<table>
<thead>
<tr>
<th>( i )</th>
<th>( x_l )</th>
<th>( x_u )</th>
<th>( x_r )</th>
<th>( f(x_l) )</th>
<th>( f(x_u) )</th>
<th>( f(x_r) )</th>
<th>( f(x_l) \cdot f(x_r) )</th>
<th>( \varepsilon_t )</th>
<th>( \varepsilon_a )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>5</td>
<td>10</td>
<td>7.5</td>
<td>2.5</td>
<td>-30.50</td>
<td>-10.25</td>
<td>-25.625</td>
<td>33.25</td>
<td>-</td>
</tr>
<tr>
<td>2</td>
<td>5</td>
<td>7.5</td>
<td>6.25</td>
<td>2.5</td>
<td>-10.25</td>
<td>-2.9375</td>
<td>-7.3438</td>
<td>11.04</td>
<td>20.00</td>
</tr>
<tr>
<td>3</td>
<td>5</td>
<td>6.25</td>
<td>5.625</td>
<td>2.5</td>
<td>-2.9375</td>
<td>0.0156</td>
<td>-0.0391</td>
<td>0.06</td>
<td>11.11</td>
</tr>
</tbody>
</table>

Therefore, after three iterations the approximate root of \( f(x) \) is \( x_r = 5.6250 \) with \( \varepsilon_t = 0.06\% \) and \( \varepsilon_a = 11.11\% \).
FALSE POSITION METHOD (Cont.)

- It is an improvement of the Bisection method.
- The bisection method converges slowly due to its behavior in redefined the size of interval that containing the root.
- The procedure begins by finding an initial interval \([x_l, x_u]\) that bracket the root.
- \(f(x_l)\) and \(f(x_u)\) are then connected using a straight line.
- The estimated root, \(x_r\) is the \(x\)-value where the straight line crosses \(x\)-axis.
- **Figure 6** indicates the graphical illustration of False Position method.

Figure 6: Graphical Illustration of False Position Method
FALSE POSITION METHOD (Cont.)

False Position Method Formula

Straight line joining the two points \((x_l, f(x_l))\) and \((x_u, f(x_u))\) is given by

\[
\frac{f(x_u) - f(x_l)}{x_u - x_l} = \frac{y - f(x_u)}{x - x_u}
\]

Since the line intersect the \(x\)-axis at \(x_r\), so for \(x = x_r, y = 0\), the following is obtained

\[
x_r - x_u = -\frac{f(x_u)(x_u - x_l)}{f(x_u) - f(x_l)}
\]

Rearranging the second equation yields the False Position Method Formula

\[
x_r = x_u - \left[\frac{f(x_u)(x_l - x_u)}{f(x_l) - f(x_u)}\right]
\]
FALSE POSITION METHOD (Cont.)

Algorithm

For the continuous equation of one variable, $f(x) = 0$, 

**Step 1:** Choose the lower guess, $x_l$ and the upper guess, $x_u$ that bracket the root such that the function has opposite sign over the interval, $x_l \leq x \leq x_u$. 

**Step 2:** The estimation root, $x_r$ is computed by using

$$x_r = x_u - \frac{f(x_u)(x_l - x_u)}{f(x_l) - f(x_u)}$$

**Step 3:** Use the following evaluations to identify the subinterval that the root lies

- If $f(x_l) \cdot f(x_r) < 0$, then the root lies in the lower subinterval. Therefore, set $x_u = x_r$ and repeat **Step 2**.
- If $f(x_l) \cdot f(x_r) > 0$, then the root lies in the upper subinterval. Therefore set $x_l = x_r$ and repeat **Step 2**.
- If $f(x_l) \cdot f(x_r) = 0$, then the root is equal to $x_r$. Terminate the computation.

**Step 4:** Calculate the approximate percent relative error,

$$\varepsilon_a = \left| \frac{x_r^{\text{present}} - x_r^{\text{previous}}}{x_r^{\text{present}}} \right| \times 100\%$$

**Step 5:** Compare with. If $\varepsilon_a < \varepsilon_s$, then stop the computation. Otherwise go to **Step 2** and repeat the process by using the new interval.
FALSE POSITION METHOD (Cont.)

Example 8

Determine the first root \( f(x) = -3x^3 + 19x^2 - 20x - 13 \) by using False position method. Use the initial guesses of \( x_l = -1 \) and \( x_u = 0 \) with stopping criterion, \( \varepsilon_s = 1\% \).

Solution

First iteration, \( x \in [-1, 0] \)

\[
\begin{align*}
  f(-1) &= 29 \\
  f(0) &= -13
\end{align*}
\]

First estimate using False position method is

\[
 x_r = 0 - \frac{(-13)(-1 - 0)}{29 - (-13)} = -0.3095
\]

\[
 f(-0.3095) = -4.9010
\]

Since \( f(x_l) \cdot f(x_r) < 0 \), the root lies in the lower subinterval. Then set \( x_u = -0.3095 \).

\[
 \varepsilon_a = \quad
\]
FALSE POSITION METHOD (Cont.)

Solution

Second iteration, \( x \in [-1, -0.3095] \).

Second estimate is

\[
x_r = -0.3095 - \frac{(-4.9010)(-1 + 0.3095)}{29 - (-4.9010)} = -0.4093
\]

\( f(-0.4093) = -1.4253 \)

Since \( f(x_l) \cdot f(x_r) < 0 \), the root lies in the lower subinterval. Then set \( x_u = -0.4093 \).

\( \varepsilon_a = 24.38\% \)
FALSE POSITION METHOD (Cont.)

Solution (Cont.)

Continue the third iteration for $x \in [-1, -0.4093]$. The results are summarized in the following table.

<table>
<thead>
<tr>
<th>$i$</th>
<th>$x_l$</th>
<th>$x_u$</th>
<th>$x_r$</th>
<th>$f(x_l)$</th>
<th>$f(x_u)$</th>
<th>$f(x_r)$</th>
<th>$f(x_l) \cdot f(x_r)$</th>
<th>$\varepsilon_a$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>-1</td>
<td>0</td>
<td>-0.3095</td>
<td>29</td>
<td>-13</td>
<td>-4.0910</td>
<td>-142.1290</td>
<td>-</td>
</tr>
<tr>
<td>2</td>
<td>-1</td>
<td>-0.3095</td>
<td>-0.4093</td>
<td>29</td>
<td>-4.9003</td>
<td>-1.4253</td>
<td>-41.3337</td>
<td>24.38</td>
</tr>
<tr>
<td>3</td>
<td>-1</td>
<td>-0.4093</td>
<td>-0.4370</td>
<td>29</td>
<td>-1.4241</td>
<td>-0.3812</td>
<td>-11.0548</td>
<td>6.33</td>
</tr>
<tr>
<td>4</td>
<td>-1</td>
<td>-0.4370</td>
<td>-0.4443</td>
<td>29</td>
<td>-0.3820</td>
<td>-0.1002</td>
<td>-2.0907</td>
<td>1.65</td>
</tr>
<tr>
<td>5</td>
<td>-1</td>
<td>-0.4443</td>
<td>-0.4462</td>
<td>29</td>
<td>-0.1002</td>
<td>-0.0267</td>
<td>-0.7743</td>
<td>0.43</td>
</tr>
</tbody>
</table>

Therefore, after fifth iterations the approximate root of $f(x)$ is $x_r = -0.4462$ with $\varepsilon_a = 0.43\%$. 
OPEN METHODS

- The idea of this method is to consider at least one initial guess which is not necessarily bracket the root.
- Normally, the chosen initial value(s) must be close to the actual root that can be found by plotting the given function against its independent variable.
- In every step of root improvement, $x_r$ of previous step is considered as the previous value for the present step.
- In general, open methods provides no guarantee of convergence to the true value, but once it is converge, it will converge faster than bracketing methods.

Two Types of Methods

1. Secant Method
2. Newton Raphson Method
NEWTON RAPHSON METHOD

- It is an open method for finding roots of $f(x) = 0$ by using the successive slope of the tangent line.
- The Newton Raphson method is applicable if $f(x)$ is continuous and differentiable.
- **Figure 6** shows the graphical illustration of Newton Raphson method.
- Numerical scheme starts by choosing the initial point, $x_0$ as the first estimation of the solution.
- The improvement of the estimation of $x_1$ is obtained by taking the tangent line to $f(x)$ at the point $(x_0, f(x_0))$ and extrapolate the tangent line to find the point of intersection with an $x$–axis.

![Graphical Illustration of Newton Raphson Method](image-url)
NEWTON RAPHSON METHOD (Cont.)

**Slope for the first iteration is:**

\[ f'(x_0) = \frac{f(x_0) - 0}{x_0 - x_1} \]  

\[ x_1 = x_0 - \frac{f(x_0)}{f'(x_0)} \]

**Rearranging equation (1) yields:**

- The next estimation, \( x_2 \) is the intersection of the tangent line \( f(x) \) at the point \((x_1, f(x_1))\).
- The estimation, \( x_{i+1} \) is the intersection of the tangent line \( f(x) \) at the point \((x_i, f(x_i))\). The slope of the \( i^{th} \) iteration is

\[ f'(x_i) = \frac{f(x_i) - 0}{x_i - x_{i+1}} \]  

\[ x_{i+1} = x_i - \frac{f(x_i)}{f'(x_i)} \]

**Rearranging equation (2) gives Newton Raphson Formula:**
NEWTON RAPHSON METHOD (Cont.)

Algorithm

For the continuous and differentiable function, \( f(x) = 0 \):

**Step 1:** Choose initial value, \( x_0 \) and find \( f'(x_0) \).

**Step 2:** Compute the next estimate, \( x_{i+1} \) by using Newton Raphson formula

\[
x_{i+1} = x_i - \frac{f(x_i)}{f'(x_i)}
\]

**Step 3:** Calculate the approximate percent relative error, \( \varepsilon_a \)

\[
\varepsilon_a = \left| \frac{x_{i+1} - x_i}{x_{i+1}} \right| \times 100\%
\]

**Step 4:** Compare \( \varepsilon_s \) with \( \varepsilon_a \). If \( \varepsilon_a < \varepsilon_s \), the computation is stopped. Otherwise, repeat **Step 2**.
NEWTON RAPHSON METHOD (Cont.)

Example 8

Determine the first root \( f(x) = 8e^{-x} \sin(x) - 1 \) by using Newton Rapshon method. Use the initial guesses of \( x_0 = 0.3 \) and perform the computation up to three iterations. (Use radian mode in your calculator)

Solution

\[
f(x) = 8e^{-x} \sin(x) - 1
\]

\[
f'(x) = 8e^{-x} (\cos(x) - \sin(x))
\]

First iteration, \( x_0 = 0.3 \)

\[
f(0.3) = 8e^{-0.3} \sin(0.3) - 1 = 0.7514,
\]

\[
f'(0.3) = 8e^{-0.3} (\cos(0.3) - \sin(0.3)) = 3.9104,
\]

\[
x_1 = x_0 - \frac{f(x_0)}{f'(x_0)} = 0.3 - \frac{0.7514}{3.9104} = 0.1078
\]

\[
\varepsilon_a = \left| \frac{0.1078 - 0.3}{0.1078} \right| \times 100\% = 178.18\%
\]
NEWTON RAPHSON METHOD (Cont.)

Solution (Cont.)

Continue the second iteration and the results are summarised as follows.

<table>
<thead>
<tr>
<th>No. of iteration</th>
<th>$i$</th>
<th>$x_i$</th>
<th>$f(x_i)$</th>
<th>$f'(x_i)$</th>
<th>$x_{i+1}$</th>
<th>$\varepsilon_a \ (%)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0</td>
<td>0.3</td>
<td>0.7514</td>
<td>3.9104</td>
<td>0.1078</td>
<td>178.18</td>
</tr>
<tr>
<td>2</td>
<td>1</td>
<td>0.1078</td>
<td>-0.2270</td>
<td>6.3674</td>
<td>0.1435</td>
<td>24.84</td>
</tr>
<tr>
<td>3</td>
<td>2</td>
<td>0.1435</td>
<td>-0.0090</td>
<td>5.8684</td>
<td>0.1450</td>
<td>1.05</td>
</tr>
</tbody>
</table>

Therefore, after three iterations the approximated root of $f(x)$ is $x_3 = 0.1450$ with $\varepsilon_a = 1.05\%$. 
NEWTON RAPHSON METHOD (Cont.)

Pitfalls of the Newton Raphson Method

**Case 1:** The tendency of the results obtained from the Newton Raphson method to oscillate around the local maximum or minimum without converge to the actual root.

**Case 2:** Division by zero involve in the Newton Raphson formula when $f'(x) = 0$. 
NEWTON RAPHSON METHOD (Cont.)

Pitfalls of the Newton Raphson Method

Case 3: In some cases where the function $f(x)$ is oscillating and has a number of roots, one may choose an initial guess close to a root. The guesses may jump and converge to some other roots and the process become oscillatory, which leads to endless cycle of fluctuations between $x_i$ and $x_{i+1}$ without converge to the desired root.
SECANT METHOD

Introduction

- In many cases, the derivative of a function is very difficult to find or even is not differentiable.
- Alternative approach is by using secant method.

- The slope in Newton’s Rapshon method is substituted with backward finite divided difference

\[
    f'(x_i) = \frac{f(x_{i-1}) - f(x_i)}{x_{i-1} - x_i}
\]

- The secant method formula is:

\[
    x_{i+1} = x_i - \left[ \frac{f(x_i)(x_{i-1} - x_i)}{f(x_{i-1}) - f(x_i)} \right]
\]

Figure 7: Graphical Illustration of Secant Method
For the continuous function, \( f(x) = 0 \):

**Step 1:** Choose initial values, \( x_{-1} \) and \( x_0 \). Find \( f(x_{-1}) \) and \( f(x_0) \).

**Step 2:** Compute the next estimate, \( x_{i+1} \) by using secant method formula

\[
x_{i+1} = x_i - \frac{f(x_i)(x_{i-1} - x_i)}{f(x_{i-1}) - f(x_i)}
\]

**Step 3:** Calculate the approximate percent relative error, \( \varepsilon_a \)

\[
\varepsilon_a = \left| \frac{x_{i+1} - x_i}{x_{i+1}} \right| \times 100\%
\]

**Step 4:** Compare \( \varepsilon_s \) with \( \varepsilon_a \). If \( \varepsilon_a < \varepsilon_s \), the computation is stopped. Otherwise, repeat **Step 2**.
SECANT METHOD (Cont.)

Example 9

Determine one of the real root(s) of \( f(x) = -12 - 21x + 18x^2 - 2.4x^3 \) by using secant method with initial guesses of \( x_{-1} = 1.0 \) and \( x_0 = 1.3 \). Perform the computation until \( \varepsilon_a < 5\% \).

Solution

First iteration, \( x_{-1} = 1.0 \) and \( x_0 = 1.3 \)

\[ f(1.0) = -17.4 \]

\[ f(1.3) = -14.528 \]

\[ x_1 = x_0 - \frac{f(x_0)(x_{-1} - x_0)}{f(x_{-1}) - f(x_0)} \]

\[ = 1.3 - \frac{-14.1528(1-1.3)}{-17.4 + 14.1528} = 2.6075 \]

\[ \varepsilon_a = \left| \frac{2.6075 - 1.3}{2.6075} \right| \times 100\% = 50.14\% > \varepsilon_s \]
SECANT METHOD (Cont.)

Solution (Cont.)

Continue the second iteration and the results are summarised as follows.

<table>
<thead>
<tr>
<th>No. of Iteration</th>
<th>i</th>
<th>$x_{i-1}$</th>
<th>$x_i$</th>
<th>$f(x_{i-1})$</th>
<th>$f(x_i)$</th>
<th>$x_{i+1}$</th>
<th>$\varepsilon_a(%)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0</td>
<td>1</td>
<td>1.3</td>
<td>-17.4</td>
<td>-14.152</td>
<td>2.6075</td>
<td>50.14</td>
</tr>
<tr>
<td>2</td>
<td>1</td>
<td>1.3</td>
<td>2.6075</td>
<td>-14.1528</td>
<td>13.0780</td>
<td>1.9796</td>
<td>31.72</td>
</tr>
<tr>
<td>3</td>
<td>2</td>
<td>2.6075</td>
<td>1.9796</td>
<td>13.0780</td>
<td>-1.6519</td>
<td>2.0500</td>
<td>3.44</td>
</tr>
</tbody>
</table>

Therefore, after three iterations the approximated root of $f(x)$ is $x_3 = 2.0500$ with $\varepsilon_a = 3.44\%$. 
## Conclusion

<table>
<thead>
<tr>
<th>Bracketing Method</th>
<th>Open Method</th>
</tr>
</thead>
<tbody>
<tr>
<td>Need two initial guesses</td>
<td>Can involve one or more initial guesses</td>
</tr>
<tr>
<td>The root is located within an interval prescribed by a lower and an upper bound.</td>
<td>Not necessarily bracket the root.</td>
</tr>
<tr>
<td>Always work but converge slowly</td>
<td>Do not always work (can diverge) but when they do they usually converge much more quickly.</td>
</tr>
</tbody>
</table>
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