THEORY OF STRUCTURES
CHAPTER 2 : DEFLECTION
(UNIT LOAD METHOD)
PART 2

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Chapter 2 : Part 2 – Unit Load Method

• Aims
  – Determine the slope and deflection by using Unit Load Method

• Expected Outcomes :
  – Able to analyze determinate beam – deflection and slope by Unit Load Method

• References
  – Structural Analysis, SI Edition by Aslam Kassimali, Cengage Learning
  – Structural Analysis, Coates, Coatie and Kong
Principle of Virtual Work

• The internal work in transversely loaded beams is taken equal to the strain energy due to bending moment.
• The virtual force $F_i$ in the $i^{th}$ mass element in $\Delta = F^* \cdot e_i$ may be taken equal to the bending moment $m_{ij}$ in the $i^{th}$ mass element due to a unit load at coordinate $j$. 
Principle of Virtual Work (Displacement)

- Sometimes referred as the Unit-Load Method
- Generally provides of obtaining the displacement and slope at a specific point on structure i.e. beam, frame or truss
- In general, the principle states that:

\[
\sum P\Delta = \sum u\delta
\]
Principle of Virtual Work (Displacement)

• Consider the structure (or body) to be of arbitrary shape

• Suppose it is necessary to determine the displacement $\Delta$ of point A on the body caused by the “real loads” $P_1$, $P_2$ and $P_3$
Principle of Virtual Work (Displacement)

• Since no external load acts on the body at A and in the direction of the displacement $\Delta$, the displacement can be determined by first placing on the body a “virtual” load such that this force $P'$ acts in the same direction as $\Delta$, (see Figure)
Principle of Virtual Work (Displacement)

- We will choose $P'$ to have a unit magnitude, $P' = 1$
- Once the virtual loadings are applied, then the body is subjected to the real loads $P_1$, $P_2$ and $P_3$, (see Figure)
- Point A will be displaced an amount $\Delta$ causing the element to deform an amount $dL$
Principle of Virtual Work (Displacement)

- As a result, the external virtual force $P'$ & internal load $u$ "ride along" by $\Delta$ and $dL$ and therefore, perform external virtual work of 1. $\Delta$ on the body and internal virtual work of $u.dL$ on the element.

\[
\Delta = \sum (u.dL)
\]

- By choosing $P' = 1$, it can be seen from the solution for $\Delta$ follows directly since $\Delta = \sum udL$. 
Principle of Virtual Work (Slope)

- A virtual couple moment \( M' \) having a unit magnitude is applied at this point.
- This couple moment causes a virtual load \( u_\theta \) in one of the elements of the body.
Principle of Virtual Work (Slope)

- Assuming that the real loads deform the element an amount $dL$, the rotation $\theta$ can be found from the virtual-work equation.
Principle of Virtual Work (Slope)

\[ \theta = \sum u.dL \]

Virtual loadings \quad Real displacement
PRINCIPLE OF UNIT LOAD METHOD

REAL LOAD

VIRTUAL LOAD

1 UNIT
• The element deform or rotate $d\theta = (M / EI) \, dx$

• The external virtual work done by the unit load is 1.

• The internal virtual work done by the moment, $m$

\[ m \, d\theta = m(M/\EI) \, dx \]

Similarly

\[ 1 \cdot \Delta = \int_{0}^{L} \frac{mM}{EI} \, dx \]

Similarly

\[ 1 \cdot \theta = \int_{0}^{L} \frac{mM}{EI} \, dx \]
Example 1

Determine the displacement at point B of the steel beam shown in figure.

Take $E = 200\text{GPa}$, $I = 500 \times 10^6 \text{ mm}^4$
Real Moment, $M$

LHS

RHS

12 kN/m

10 m

$M$

$V$

$M$

$V$
Real Moment, $M$

True Moment, $M$

$12 \text{kN/m}$

$x$

$$\sum M_x = 0 \ (\text{clockwise} + \text{ve})$$

$$M + \frac{12x^2}{2} = 0$$

$$M = -6x^2$$
Virtual Moment, $m$
Virtual Moment, \( m \)

Considered RHS \( 0 < x < 10 \)

\[
\sum M_x = 0 \quad (\text{clockwise +ve})
\]

\[
m + 1.x = 0
\]

\[
m = -1.x
\]
Virtual-Work Equation

\[ 1 \text{kN} \Delta = \int_{0}^{L} \frac{mM}{EI} dx \]

\[ = \int_{0}^{10} \frac{(-x)(-6x^2)}{EI} dx \]

\[ = \frac{15 \times 10^3 \text{kN} \cdot \text{m}^3}{EI} \]

OR

\[ \Delta_B = \frac{15 \times 10^3 \text{kN} \cdot \text{m}^3}{200(10^6) \text{ kN} / \text{m}^2 \cdot (500(10^6) \text{mm}^4) \cdot (10^{-12} \text{ m}^4 / \text{mm}^4)} \]

\[ = 0.150m = 150mm \]
Determine the displacement at D of the steel beam in figure. Take $E = 200\text{GPa}$, $I = 300\text{E}6\ \text{mm}^4$
Real Moment, $M$

- $M_1$: 120 kN.m
- $M_2$: 30 kN
- $M_3$: 4.5 m

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Determine the reaction at support...

\[ \sum M_A = 0 \quad (\text{clockwise} + \text{ve}) \]
\[ 120 + 30(3) - V_B (6) = 0 \]
\[ V_B = 35kN \]

\[ \sum F_y = 0 \quad (\text{upward} + \text{ve}) \]
\[ -V_A + 35 - 30 = 0 \]
\[ V_A = 5kN \]
Member AB : LHS

120 kN.m

\[ M_1 \]

5 kN

\[ 0 \leq x \leq 3 \]

\[ \sum M_x = 0 \ (clockwise \ + \ ve) \]

\[ -M_1 - 5(x) + 120 = 0 \]

\[ M_1 = 120 - 5x \]

Member BC : RHS

\[ M_2 \]

35 kN

\[ 0 \leq x \leq 3 \]

\[ \sum M_x = 0 \ (clockwise \ + \ ve) \]

\[ M_2 - 35(x) = 0 \]

\[ M_2 = 35(x) \]
Member CD : RHS

$0 \leq x \leq 4.5$

$M_3$ clockwise

$\sum M_x = 0 \ (clockwise + ve)$

$M_3 = 0$
Determine the reaction at support

\[ \sum M_A = 0 \quad (\text{clockwise} + \text{ve}) \]

\[ 1(10.5) - V_B(6) = 0 \]

\[ V_B = 1.75\text{kN} \]

\[ \sum F_y = 0 \quad (\text{upward} + \text{ve}) \]

\[ -V_A + 1.75 - 1 = 0 \]

\[ V_A = 0.75\text{kN} \]
Member AB : LHS

0 \leq x \leq 3

\begin{align*}
\sum M_x &= 0 \quad (\text{clockwise } + \text{ve}) \\
-m_1 - 0.75(x) &= 0 \\
m_1 &= -0.75x
\end{align*}

Member BC : RHS

0 \leq x \leq 3

\begin{align*}
\sum M_x &= 0 \quad (\text{clockwise } + \text{ve}) \\
m_2 - 1.75(x) + 1(x + 4.5) &= 0 \\
m_2 &= 0.75(x) - 4.5
\end{align*}
Member CD : RHS

0 \leq x \leq 4.5

\sum M_x = 0 \ (clockwise + ve)

m_3 + 1 \cdot x = 0

m_3 = -x
Virtual-Work Equation

\[ 1 \text{kN} \Delta_d = \int_0^L \frac{mM}{EI} \, dx \]

\[ = \int_0^3 \frac{m_1 M_1}{EI} \, dx + \int_0^3 \frac{m_2 M_2}{EI} \, dx + \int_0^{4.5} \frac{m_3 M_3}{EI} \, dx \]

\[ = \int_0^3 \frac{(-0.75x)(120 - 5x)}{EI} \, dx + \int_0^3 \frac{(0.75x - 4.5)(35x)}{EI} \, dx \]

\[ + \int_0^{4.5} \frac{(-x)(0)}{EI} \, dx \]
\[
\Delta_D = -\frac{371.25}{EI} - \frac{472.5}{EI} + \frac{0}{EI} = -\frac{843.75 \text{ kN} \cdot \text{m}^3}{EI}
\]

\[
\Delta_D = \frac{-843.75 \text{ kN} \cdot \text{m}^3}{200 \times (10^6) \text{ kN} / \text{m}^2 \times (300 \times (10^6) \text{ mm}^4) \times (10^{-12} \text{ m}^4 / \text{mm}^4)} = -0.0141 \text{ m} = -14.1 \text{ mm}
\]
Determine the **slope** at A and **deflection** at C in the beam shown below.
Solution

Real Load (M)

Generalized coordinates

Virtual Load ($m_\theta$): Slope

Virtual Load ($m_\Delta$): Deflection
Real Load → M ?

1. Support reaction,

\[ \sum M_A = 0 \text{(clockwise +)}, \]
\[ -R_B (6) + 9(4) + 12(3) \left( \frac{3}{2} \right) = 0 \]
\[ \therefore R_B = 15 \text{kN} \]

\[ \sum F_y \uparrow^+ = 0, \]
\[ R_A - 12(3) - 9 + 15 = 0 \]
\[ \therefore R_A = 30 \text{kN} \]
Real Load \((M) : 0 \leq x \leq 3\) (segment \(AC\))

\[ \sum M_x = 0 (\text{clockwise} +), \]

\[ M_x = 30x - 12\left(\frac{x^2}{2}\right) \]

\[ \therefore M_x = 30x - 6x^2 \ldots \ldots \ldots (i) \]
Real Load (M) : $3 \leq x \leq 4$ (segment CD)

\[ \sum M_x = 0 \text{ (clockwise +)} , \]

\[ M_x = 30x - 12(3)(x - \frac{3}{2}) \]

\[ M_x = 30x - 36x + 54 \]

\[ \therefore M_x = -6x + 54 \ldots \ldots (ii) \]
Real Load (M): $0 \leq x \leq 2$ (segment BD)

$$
\sum M_x = 0 (\text{clockwise +}),
- M_x = -15x
\therefore M_x = 15x \ldots \ldots (iii)
$$
Virtual Load, m for deflection

Apply point load \( P = 1 \)

\[
\sum M_A = 0 \text{ (clockwise +)}, \quad -R_B(6) + 1(3) = 0
\]

\[
\therefore R_B = \frac{1}{2}
\]

\[
\sum F_y \uparrow^+ = 0, \quad R_A + R_B - 1 = 0
\]

\[
\therefore R_A = \frac{1}{2}
\]
Virtual Load (m): $0 \leq x \leq 3$ (segment $AC$)

$$\sum M_x = 0 \ (clockwise +),$$

$$\therefore M_x = \frac{1}{2} x \ldots \ldots (i)$$
Virtual Load (m): $3 \leq x \leq 4$ (segment CD)

$$\sum M_x = 0 (clockwise +),$$

$$\therefore M_x = -0.5x + 3 \ldots \ldots (ii)$$
Virtual Load (m): $0 \leq x \leq 2$ (segment BD)

\[ \sum M_x = 0 \text{(clockwise +)} , \]

\[ - M_x = - \frac{1}{2} x \]

\[ \therefore M_x = \frac{1}{2} x \text{..........(iii)} \]
Deflection at C, \( \Delta_D \):

\[
\Delta_D = \int \frac{Mm}{EI} \, dx
\]

\[
= \frac{1}{EI} \int_0^3 (30x - 6x^2)(0.5x) \, dx + \frac{1}{EI} \int_3^4 (-6x + 54)(-0.5x + 3)x \, dx
\]

\[
= \frac{1}{EI} \int_0^3 (15x)(0.5x) \, dx
\]

\[
= \frac{135.75}{EI}
\]
Virtual Load, m for rotation

Apply \( m_\theta = 1 \)

\[
\sum M_A = 0 \text{(clockwise +)}, \quad -R_B (6) + 1 = 0
\]

\[ R_B = \frac{1}{6} \]

\[
\sum F_y \uparrow^+ = 0, \quad R_A + R_B = 0
\]

\[ R_A = -\frac{1}{6} \]
Virtual Load (m) : $0 \leq x \leq 3$ (segment AC)

$$\sum M_x = 0 (clockwise +),$$

$$\therefore M_x = 1 - \frac{1}{6}x \quad \ldots \ldots \quad (i)$$
Virtual Load \((m)\): \(3 \leq x \leq 4\) (segment \(CD\))

\[
\sum M_x = 0 \text{ (clockwise +)},
\]

\[
\therefore M_x = 1 - \frac{1}{6} x \quad \ldots \ldots \quad (ii)
\]
Virtual Load \((m)\): \(0 \leq x \leq 2\) (segment \(BD\))

\[
\sum M_x = 0 \text{ (clockwise +)},
\]

\[-M_x = -\frac{1}{6} x\]

\[
\therefore M_x = \frac{1}{6} x \ldots \ldots (iii)
\]
Slope at A, $\theta_A$:

$$\theta_A = \int \frac{Mm}{EI} \, dx$$

$$= \frac{1}{EI} \int_0^3 (30x - 6x^2) \left(1 - \frac{x}{6}\right) \, dx + \frac{1}{EI} \int_3^4 (-6x + 54) \left(1 - \frac{x}{6}\right) \, dx$$

$$= \frac{1}{EI} \int_0^2 (15x) \left(\frac{x}{6}\right) \, dx$$

$$= 76.75 \frac{EI}{EI}$$
Determine the slope and deflection at B in the beam shown below. Given $E=200 \text{kN/mm}^2$

$\begin{align*}
I_{AB} &= 4 \times 10^6 \text{ mm}^4 \\
I_{BC} &= 8 \times 10^6 \text{ mm}^4 
\end{align*}$
Moment equation (deflection):

<table>
<thead>
<tr>
<th>Segment</th>
<th>Condition</th>
<th>$I \text{ mm}^4$</th>
<th>$m$ (deflection)</th>
<th>$M$</th>
</tr>
</thead>
<tbody>
<tr>
<td>AD</td>
<td>$0&lt;x&lt;0.5$</td>
<td>$4 \times 10^6$</td>
<td>0</td>
<td>$8x$</td>
</tr>
<tr>
<td>DB</td>
<td>$0.5&lt;x&lt;1$</td>
<td>$4 \times 10^6$</td>
<td>0</td>
<td>$8x - 2.5 (x - 0.5)^2$</td>
</tr>
<tr>
<td>BC</td>
<td>$1&lt;x&lt;3$</td>
<td>$8 \times 10^6$</td>
<td>$x - 1$</td>
<td>$8x - 2.5 (x - 0.5)^2$</td>
</tr>
</tbody>
</table>
Deflection, $\Delta_B$

\[
\Delta_B = \int \frac{mM}{EI} \, dx \\
= \int_1^3 \frac{(x - 1)(-2.5x^2 + 10.5x - 0.625)}{(200 \times 10^6)(8 \times 10^{-6})} \, dx \\
= \frac{1}{1600} \left[ -\frac{2.5x^4}{4} + \frac{13x^3}{3} - \frac{11.125x^2}{2} + 0.625x \right]_1 \\
= 0.012m \\
= 12mm
\]
Moment equation (slope):

<table>
<thead>
<tr>
<th>Segment</th>
<th>Condition</th>
<th>( l \text{ mm}^4 )</th>
<th>( m \text{ (slope)} )</th>
<th>( M )</th>
</tr>
</thead>
<tbody>
<tr>
<td>AD</td>
<td>0&lt;x&lt;0.5</td>
<td>( 4 \times 10^6 )</td>
<td>0</td>
<td>( 8x )</td>
</tr>
<tr>
<td>DB</td>
<td>0.5&lt;x&lt;1</td>
<td>( 4 \times 10^6 )</td>
<td>0</td>
<td>( 8x - 2.5 (x - 0.5)^2 )</td>
</tr>
<tr>
<td>BC</td>
<td>1&lt;x&lt;3</td>
<td>( 8 \times 10^6 )</td>
<td>–1</td>
<td>( 8x - 2.5 (x- 0.5)^2 )</td>
</tr>
</tbody>
</table>
\[
\theta_B = \int \frac{mM}{EI} \, dx \\
= \int_1^3 \frac{(1)(-2.5x^2 + 10.5x - 0.625)}{(200 \times 10^6)(8 \times 10^{-6})} \, dx \\
= \frac{1}{1600} \left[ -\frac{2.5x^3}{3} + \frac{10.5x^2}{2} - 0.625x \right]_1^3 \\
= \frac{19.1}{1600} \\
= 0.0119 \text{rad}
\]
THANKS
Author Information

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